Tutorial 5, Feb 11, 2026

- Consider $x \in \mathcal{N}(\mu_x, \sigma_x^2)$ and $y = f(x) = (x+b)^2$
  - Show that $\mu_y = (\mu_x + b)^2 + \sigma_x^2$
    * Rewrite $x = \mu_x + \delta x$ where $\delta x \sim N(0, \sigma_x^2)$
    * $\mu_y = E[y]$
      $$\begin{aligned} &= E[(x+b)^2] \\ &= E[(\mu_x + \delta x + b)^2] \\ &= E[(\mu_x + b)^2 + 2(\mu_x + b)\delta x + \delta x^2] \\ &= (\mu_x + b)^2 + 2(\mu_x + b)E[\delta x] + E[(0 - \delta x)(0 - \delta x)] \\ &= (\mu_x + b)^2 + \sigma_x^2 \end{aligned}$$
      - Note that $E[\delta x] = 0$ since it's zero mean, and $E[\delta x^2]$ is the same as the variance of δx which is σ_x^2 by definition
    * Notice that we've reached an interesting conclusion that the mean of y is dependent on the variance of x
  - Find $\bar{\mu}$ via linearization and show that it is biased
    * $\dfrac{\partial f}{\partial x} = 2(x+b)$
    * $y \approx f(\mu_x) + \left.\dfrac{\partial f}{\partial x}\right|_{x=\mu_x} \delta x = (\mu_x + b)^2 + 2(\mu_x + b)\delta x = y_{\text{lin}}$
    * $\bar{\mu}_y = E[y_{\text{lin}}] = (\mu_x + b)^2$ once again since $E[\delta x] = 0$
    * Notice that $\mu_y - \bar{\mu}_y = \sigma_x^2$, so our linearized mean has a bias and we don't recover the true mean
- x, y are uncorrelated if $E[xy] = E[x]E[y]$; starting from the definition of σ_{xy}, show that this is equivalent to $\sigma_{xy} = 0$
  - Let $E[x] = \mu_x, E[y] = \mu_y$
  - $\sigma_{xy} = E[(x - \mu_x)(y - \mu_y)]$
    $$\begin{aligned} &= E[xy - \mu_x y - \mu_y x - \mu_x \mu_y] \\ &= E[xy] - \mu_x E[y] - \mu_y E[x] + \mu_x \mu_y \\ &= E[xy] - \mu_x \mu_y - \mu_y \mu_x + \mu_x \mu_y \\ &= E[xy] - \mu_x \mu_y \\ &= 0 \end{aligned}$$