Lecture 5, Jan 12, 2026

Vehicle Models

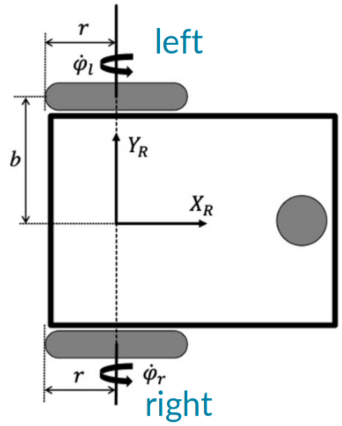

Figure 1: Differential drive robot.

- Consider the unicycle model kinematics we derived previously for differential drive (note we redefine b to be half the distance between wheels) $\begin{bmatrix} v \\ \omega \end{bmatrix} = \frac{1}{2}\begin{bmatrix} r & r \\ r/2b & -r/2b \end{bmatrix}\begin{bmatrix} \dot{\varphi}_r \\ \dot{\varphi}_l \end{bmatrix}$
  - Define the inertial frame configuration $\boldsymbol{q} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$
  - To handle the nonholonomic constraint, consider $\dot{\boldsymbol{q}}$ in the vehicle frame, since we are constrained to have $u = 0$
    * $\dot{\boldsymbol{q}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v \\ 0 \\ \omega \end{bmatrix}$
  - The overall kinematics in inertial frame is $\dot{\boldsymbol{q}} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} r/2 & r/2 \\ r/2b & r/2b \end{bmatrix}\begin{bmatrix} \dot{\varphi}_r \\ \dot{\varphi}_l \end{bmatrix} = \boldsymbol{G}(q)\boldsymbol{p}$
- Generally we can write the nonholonomic constraints in matrix form as $\boldsymbol{H}(q)^T\dot{\boldsymbol{q}} = 0$, so the admissible velocities consists of the null space of $\boldsymbol{H}(\boldsymbol{q})^T$
  - Let the generalized velocity $\boldsymbol{p}$, then the null space of $\boldsymbol{H}(\boldsymbol{q})^T$ is $\dot{\boldsymbol{q}} = \boldsymbol{G}(\boldsymbol{q})\boldsymbol{p}$ where $\boldsymbol{H}(\boldsymbol{q})^T\boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{0}$
  - For the unicycle model, $\boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix}$
- To model the vehicle dynamics as well, we usually use Euler-Lagrange: $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial^T L}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial^T L}{\partial \boldsymbol{q}} + \boldsymbol{\tau} + \boldsymbol{H}(\boldsymbol{q})\boldsymbol{\lambda}$ where the $\boldsymbol{H}(\boldsymbol{q})\boldsymbol{\lambda}$ models nonholonomic constraints, with $\boldsymbol{\lambda}$ being the Lagrange multipliers
  - The generalized forces can be expressed as $\boldsymbol{\tau} = \boldsymbol{G}(\boldsymbol{q})\boldsymbol{\nu}$ so that the nonholonomic constraints are satisfied
- For the unicycle model:
  - $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2, V = 0$
  - The Lagrange multiplier in this case represents the constraint forces arising from the vehicle kinematics (i.e. force from the wheel no-slip constraints), but we don't know this force so we try to eliminate it
  - $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial^T L}{\partial \dot{\boldsymbol{q}}}\right) = \begin{bmatrix} m\ddot{x} \\ m\ddot{y} \\ I\ddot{\theta} \end{bmatrix}, \frac{\partial^T L}{\partial \boldsymbol{q}} = \boldsymbol{0}, \boldsymbol{\tau} = \boldsymbol{G}(\boldsymbol{q})\boldsymbol{\nu}$

– To eliminate the unknown Lagrange multiplier, premultiply the EL equation by $\boldsymbol{G}(\boldsymbol{q})^T$:

* $$\boldsymbol{G}(\boldsymbol{q})^T \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial^T L}{\partial \dot{\boldsymbol{q}}}\right) - \boldsymbol{G}(\boldsymbol{q})^T \frac{\partial^T L}{\partial \boldsymbol{q}} = \boldsymbol{G}(\boldsymbol{q})^T \boldsymbol{\tau} + \boldsymbol{G}(\boldsymbol{q})^T \boldsymbol{H}(\boldsymbol{q})\boldsymbol{\lambda}$$

$$\implies \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m\ddot{x} \\ m\ddot{y} \\ I\ddot{\theta} \end{bmatrix} = \boldsymbol{\nu}$$

* Note in this case $\boldsymbol{G}(q)^T \boldsymbol{G}(q) = \mathbf{1}$, but this is not true in general

– Recall the generalized velocity is $\boldsymbol{p} = \begin{bmatrix} v \\ \omega \end{bmatrix} \implies \dot{\boldsymbol{p}} = \begin{bmatrix} \ddot{x}\cos\theta + \ddot{y}\sin\theta \\ \ddot{\theta} \end{bmatrix}$

– Therefore $\begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\omega} \end{bmatrix} = \boldsymbol{\nu} \iff \boldsymbol{M}\dot{\boldsymbol{p}} = \boldsymbol{\nu}$

– The complete model is $\begin{bmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \boldsymbol{G}(\boldsymbol{q}) \\ \mathbf{0} & -\boldsymbol{M}^{-1}\boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \boldsymbol{M}^{-1} \end{bmatrix} \boldsymbol{u}$

* This uses a damped model for forces $\boldsymbol{\nu} = -\boldsymbol{D}\boldsymbol{p} + \boldsymbol{u}$ where $\boldsymbol{u} = \begin{bmatrix} f \\ g \end{bmatrix}$, f is some longitudinal thrust and g is some steering torque

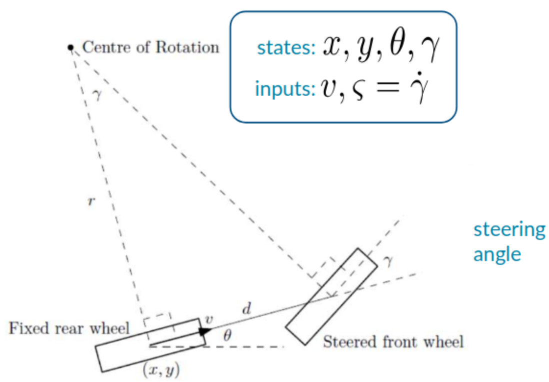

Figure 2: Bicycle model derivation.

- For the bicycle model:
  – Define state $\begin{bmatrix} x \\ y \\ \theta \\ \gamma \end{bmatrix}$, generalized velocity $\boldsymbol{p} = \begin{bmatrix} v \\ \varsigma \end{bmatrix}$ (where ς is the steering velocity)

  – Form the nonholonomic constraints by taking lateral slip constraints from the wheel model for both wheels, and find the nullspace of $\boldsymbol{H}(\boldsymbol{q})^T$ to get $\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ \frac{\tan\gamma}{d} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \varsigma \end{bmatrix} \iff \dot{\boldsymbol{q}} = \boldsymbol{G}(\boldsymbol{q})\boldsymbol{p}$

  – Form the Lagrangian: $$L = \frac{1}{2}(m_r + m_f)(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(I_r + m_f d^2)\dot{\theta}^2 + \frac{1}{2}I_f(\dot{\theta} + \dot{\gamma})^2$$
  $$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}I_f(\dot{\theta} + \dot{\gamma})^2$$

    * Note this comes from the kinetic energy of the forward and rear wheels combined

  – Now we have $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial^T L}{\partial \dot{\boldsymbol{q}}}\right) = \begin{bmatrix} m\ddot{x} \\ m\ddot{y} \\ I\ddot{\theta} + I_f(\ddot{\theta} + \ddot{\gamma}) \\ I_f(\ddot{\theta} + \ddot{\gamma}) \end{bmatrix} = \boldsymbol{M}\ddot{\boldsymbol{q}}, \frac{\partial^T L}{\partial \boldsymbol{q}} = \mathbf{0}, \boldsymbol{\tau} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{\nu} = \boldsymbol{F}(\boldsymbol{q})\boldsymbol{\nu}$

    * $\boldsymbol{M} = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & I + I_f & I_f \\ 0 & 0 & I_f & I_f \end{bmatrix}$
    * Note $\boldsymbol{\tau}$ is somewhat arbitrary here; the matrix $\boldsymbol{F}(\boldsymbol{q})$ is not necessarily related to $\boldsymbol{G}(\boldsymbol{q})$, and instead is just based on what we choose
  - Premultiply by $\boldsymbol{G}(\boldsymbol{q})^T$ to eliminate the Lagrange multiplier again to get $\dot{\boldsymbol{p}} = \boldsymbol{M}(\boldsymbol{q})^{-1}(-(\boldsymbol{G}(q)^T \boldsymbol{M}\dot{\boldsymbol{G}}(\boldsymbol{q}) + \boldsymbol{D})\boldsymbol{p} + \boldsymbol{u}) = \boldsymbol{M}(\boldsymbol{q})^{-1}(-\boldsymbol{D}(\boldsymbol{q}, \boldsymbol{p})\boldsymbol{p} + \boldsymbol{u})$
    * Note $\boldsymbol{M}(\boldsymbol{q}) = \boldsymbol{G}(\boldsymbol{q})^T \boldsymbol{M}\boldsymbol{G}(\boldsymbol{q})$
    * We again used damped forces $\boldsymbol{\nu} = -\boldsymbol{D}\boldsymbol{p} + \boldsymbol{u}$
  - Complete system model: $\begin{bmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \boldsymbol{G}(\boldsymbol{q}) \\ \mathbf{0} & -\boldsymbol{M}(\boldsymbol{q})^{-1}\boldsymbol{D}(\boldsymbol{q}, \boldsymbol{p}) \end{bmatrix} \begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \boldsymbol{M}(\boldsymbol{q})^{-1} \end{bmatrix} \boldsymbol{u}$ In general all vehicle models can be written as a nonlinear differential equation $\dot{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{B}(\boldsymbol{x})\boldsymbol{u}$ where $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{bmatrix}$ consisting of the configuration (pose, etc) and generalized velocity
  - If we ignore dynamics, then $\boldsymbol{A} = 0$
- Additional constraints can be added to the model to limit $\{\boldsymbol{x}, \boldsymbol{u}\}$ to an allowed set S_{allowed}, e.g. turning rates or obstacles
  - These constraints are in general non-convex and makes optimization much harder
- Example: imposing a curvature constraint on the unicycle model
  - Define the curvature $k = \frac{\omega}{v} = \frac{1}{b}\frac{\dot{\varphi}_r - \dot{\varphi}_l}{\dot{\varphi}_r + \dot{\varphi}_l}$ and radius of curvature $R = \frac{1}{|k|}$
  - A max curvature constraint would impose $|\frac{\omega}{v}| = |k| \leq k_{\text{max}} = \frac{1}{R_{\text{min}}}$