Lecture 4, Jan 12, 2026

Wheel Differential Kinematics

- For the standard wheel, we assume rolling without slipping, so $v_x = \dot{\varphi} r$ and $v_y, v_z = 0$

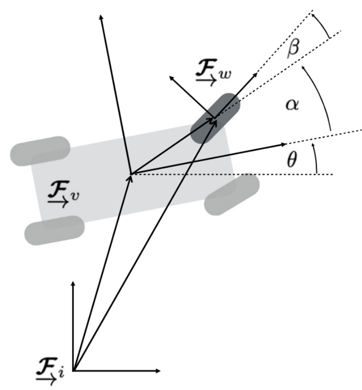

Figure 1: Derivation of the standard wheel model.

- Consider a vehicle with heading θ, with a wheel at angle α and distance $\|\underrightarrow{r}^{wv}\| = d$ relative to vehicle frame, with steering angle β
  - $\underrightarrow{r}^{wi} = \underrightarrow{r}^{vi} + \underrightarrow{r}^{wv} \implies \boldsymbol{v}_w^{wi} = \boldsymbol{v}_w^{vi} + \boldsymbol{\omega}_w^{vi\times} \boldsymbol{r}_w^{wv}$ in the wheel frame
  - This becomes $\begin{bmatrix} \dot{\varphi} r \\ 0 \\ 0 \end{bmatrix} = \boldsymbol{C}_3(\alpha + \beta)\boldsymbol{v}_v^{vi} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}^\times \begin{bmatrix} d\cos\beta \\ -d\sin\beta \\ 0 \end{bmatrix}$
  - Simplify: $\begin{bmatrix} \dot{\varphi} r \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) & 0 \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{v}_v^{vi} + \begin{bmatrix} d\sin\beta \\ d\cos\beta \\ 0 \end{bmatrix} \dot{\theta}$
    - The 3 equations express the rolling without sliding, no sideways sliding, and contact with ground constraints
  - Let $\dot{\boldsymbol{q}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}$ be the pose rate in inertial frame, so in vehicle frame $\dot{\boldsymbol{\xi}} = \begin{bmatrix} v \\ u \\ \omega \end{bmatrix} = \boldsymbol{C}_3(\theta)\dot{\boldsymbol{q}}$ (where $\omega = \dot{\theta}$)
    - $\begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) & d\sin\beta \end{bmatrix} \dot{\boldsymbol{\xi}} = \dot{\varphi} r$
    - $\begin{bmatrix} -\sin(\alpha+\beta) & \cos(\alpha+\beta) & d\cos\beta \end{bmatrix} \dot{\boldsymbol{\xi}} = 0$
- Example: differential drive model
  - $\begin{bmatrix} \cos(\alpha_r+\beta_r) & \sin(\alpha_r+\beta_r) & d_r\sin\beta_r \\ \cos(\alpha_l+\beta_l) & \sin(\alpha_l+\beta_l) & d_l\sin\beta_l \\ -\sin(\alpha_r+\beta_r) & \cos(\alpha_r+\beta_r) & d_r\cos\beta_r \\ -\sin(\alpha_l+\beta_l) & \cos(\alpha_l+\beta_l) & d_l\cos\beta_l \end{bmatrix} \begin{bmatrix} v \\ u \\ \omega \end{bmatrix} = \begin{bmatrix} \dot{\varphi}_r r \\ \dot{\varphi}_l r \\ 0 \\ 0 \end{bmatrix}$
  - Substituting and simplifying: $\begin{bmatrix} 1 & 0 & b \\ 1 & 0 & -b \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ u \\ \omega \end{bmatrix} = \begin{bmatrix} \dot{\varphi}_r r \\ \dot{\varphi}_l r \\ 0 \\ 0 \end{bmatrix}$
    - These are known as the *differential kinematics* of the robot, relating the body-centric velocity to the wheel speeds
    - e.g. If $\dot{\varphi}_r = \dot{\varphi}_l$, we get $\omega = 0$ which intuitively makes sense
  - Solving for wheel rates gives us *inverse differential kinematics*: $\begin{bmatrix} \dot{\varphi}_r \\ \dot{\varphi}_l \end{bmatrix} = \frac{1}{r}\begin{bmatrix} 1 & b \\ 1 & -b \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$

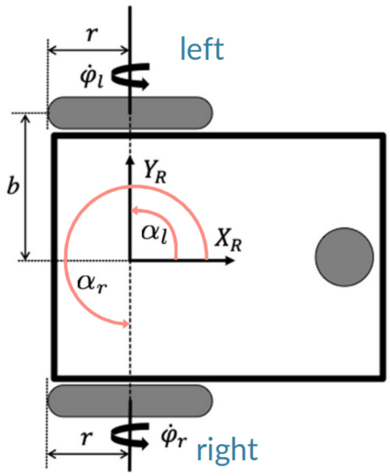

Figure 2: Differential drive robot model.

- Solving for vehicle speed gives us *forward differential kinematics*: $\begin{bmatrix} v \\ \omega \end{bmatrix} = \frac{1}{2} \begin{bmatrix} r & r \\ r/b & -r/b \end{bmatrix} \begin{bmatrix} \dot{\varphi}_r \\ \dot{\varphi}_l \end{bmatrix}$

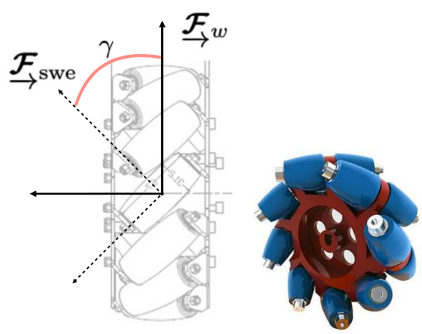

Figure 3: Swedish wheel.

- Swedish wheels (or Mecanum wheels) have rollers on the wheel which allows for sideways motion at an angle γ
  - Following the same derivation gives us the following set of constraints:
    * $\begin{bmatrix} \cos(\alpha+\beta+\gamma) & \sin(\alpha+\beta+\gamma) & d\sin(\beta+\gamma) \end{bmatrix} \dot{\boldsymbol{\xi}} = \dot{\varphi} r \cos\gamma + \dot{\varphi}_s r_s$
    * $\begin{bmatrix} -\sin(\alpha+\beta+\gamma) & \cos(\alpha+\beta+\gamma) & d\cos(\beta+\gamma) \end{bmatrix} \dot{\boldsymbol{\xi}} = -\dot{\varphi} r \sin\gamma$
    * Note we can recover the standard wheel model by simply setting $\dot{\varphi}_s = 0$
  - Note since the small wheels are passive, $\dot{\varphi}_s$ can be anything, so the first equation does not constrain the motion and just acts as another degree of freedom, i.e. we usually only have the lateral constraint

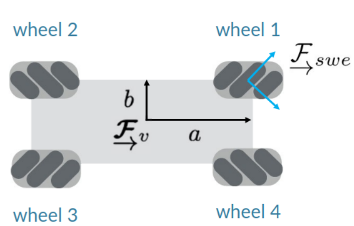

Figure 4: Swedish wheel vehicle.

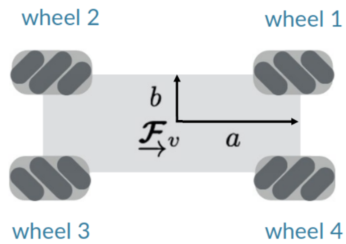

Figure 5: Swedish wheel configuration without degeneracy (bottom-up view).

- Example: vehicle with 4 Swedish wheels
  - Using the first configuration and expanding the lateral constraints (since the rollers are unconstrained):
    * $$\begin{bmatrix} 1 & 1 & -(b-a) \\ 1 & -1 & -(b-a) \\ 1 & 1 & b-a \\ 1 & -1 & b-a \end{bmatrix} \begin{bmatrix} v \\ u \\ \omega \end{bmatrix} = \begin{bmatrix} \dot{\varphi}_1 r \\ \dot{\varphi}_2 r \\ \dot{\varphi}_3 r \\ \dot{\varphi}_4 r \end{bmatrix}$$
    * Notice that in the case of $a = b$, the last column is cleared out and we no longer have control over ω; intuitively this is because when the wheels are symmetric about the centre, the vehicle can be rotated freely regardless of wheel rotation
  - Using the second configuration we can avoid the degeneracy:
    * $$\begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ \dot{\varphi}_3 \\ \dot{\varphi}_4 \end{bmatrix} = \frac{1}{r} \begin{bmatrix} 1 & -1 & -(a+b) & & & \\ 1 & 1 & -(a+b) & & & \\ 1 & -1 & (a+b) & 1 & 1 & (a+b) \end{bmatrix} \begin{bmatrix} v \\ u \\ \omega \end{bmatrix}$$
  - Since we can individually control all 4 wheels but the vehicle only has 3 degrees of freedom, the forward kinematics are not unique; we can use the psuedoinverse to recover the forward model:
    * $$\begin{bmatrix} v \\ u \\ \omega \end{bmatrix} = \frac{r}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -\frac{1}{a+b} & -\frac{1}{a+b} & \frac{1}{a+b} & \frac{1}{a+b} \end{bmatrix} \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ \dot{\varphi}_3 \\ \dot{\varphi}_4 \end{bmatrix}$$