Lecture 3, Jan 7, 2026

Mathematical Background

- There is no representation of rotations that has exactly 3 parameters (and therefore no constraints) and is also free of singularities
- Rotation representations include axis-angle (4 parameters including a unit length $\boldsymbol{a}$ and angle ϕ), quaternions (4 parameters subject to unit length), and Gibbs vector $\boldsymbol{g} = \boldsymbol{a}\tan\frac{\phi}{2}$ (3 parameters but singularity at $\phi = \pi$)
  - $\boldsymbol{\varepsilon} = \boldsymbol{a}\sin\frac{\phi}{2}, \eta = \cos\frac{\phi}{2}$ to convert between axis angle and quaternions
- For small perturbations, we have approximately $\boldsymbol{C} = 1 - \boldsymbol{\theta}^{\times}$ where $\boldsymbol{\theta} = \phi\boldsymbol{a}$
  - In this case the product of the principal rotations (Euler angles) is approximately their sum
  - This is used a lot in optimization
- Time derivatives of vectors seen in different frames is related as $\underset{\rightarrow}{r}^{\bullet} = \underset{\rightarrow}{r}^{\circ} + \underset{\rightarrow}{\omega}_{21} \times \underset{\rightarrow}{r} \iff \dot{\boldsymbol{r}}_1$
  - In terms of coordinates, $\boldsymbol{C}_{12}(\dot{\boldsymbol{r}}_2 + \boldsymbol{\omega}_2^{21\times}\boldsymbol{r}_2)$
  - We often have $\underset{\rightarrow}{r}$ denoting some position, $\underset{\rightarrow}{\mathcal{F}}_1$ being an inertial frame and $\underset{\rightarrow}{\mathcal{F}}_2$ being a moving frame (e.g. vehicle frame)
- We can show that the rotation matrix obeys Possion's equation: $\dot{\boldsymbol{C}}_{21} = -\boldsymbol{\omega}_2^{21\times}\boldsymbol{C}_{21}$
  - This means we can obtain the rotation matrix in a navigation scenario by integrating the equation, with $\boldsymbol{\omega}$ obtained from a sensor attached to the vehicle
- Consider a point P, which is $\underset{\rightarrow}{r}^{pi}$ in frame i and $\underset{\rightarrow}{r}^{pv}$ in frame v; given the pose of frame v, $\{\boldsymbol{r}_i^{vi}, \boldsymbol{C}_{iv}\}$, the vectors can be related as $\boldsymbol{r}_i^{pi} = \boldsymbol{C}_{iv}r_v^{pv} + r_i^{vi}$
  - $\boldsymbol{r}_a^{bc}$ denotes the coordinates of the vector from c to b, expressed in frame a
  - Notice that the indices of the translation $\boldsymbol{r}_i^{vi}$ look reversed, since we need to add the coordinates of frame v relative to frame i to get from v to i
- As a homogeneous transformation, $\begin{bmatrix}\boldsymbol{r}_i^{pi} \\ 1\end{bmatrix} = \begin{bmatrix}\boldsymbol{C}_{iv} & \boldsymbol{r}_i^{vi} \\ \boldsymbol{0}^T & 1\end{bmatrix}\begin{bmatrix}\boldsymbol{r}_v^{pv} \\ 1\end{bmatrix} = \boldsymbol{T}_{iv}\begin{bmatrix}\boldsymbol{r}_v^{pv} \\ 1\end{bmatrix}$
  - Note $\boldsymbol{T}_{iv}^{-1} = \begin{bmatrix}\boldsymbol{C}_{iv} & \boldsymbol{r}_i^{vi} \\ \boldsymbol{0}^T & 1\end{bmatrix}^{-1} = \begin{bmatrix}\boldsymbol{C}_{iv}^T & -\boldsymbol{C}_{iv}^T\boldsymbol{r}_i^{vi} \\ \boldsymbol{0}^T & 1\end{bmatrix} = \begin{bmatrix}\boldsymbol{C}_{vi} & -\boldsymbol{r}_v^{vi} \\ \boldsymbol{0}^T & 1\end{bmatrix} = \begin{bmatrix}\boldsymbol{C}_{vi} & \boldsymbol{r}_v^{iv} \\ \boldsymbol{0}^T & 1\end{bmatrix} = \boldsymbol{T}_{vi}$
- The generalization of angular velocity for poses is $\boldsymbol{\varpi}_v^{vi} = \begin{bmatrix}\boldsymbol{\nu}^{vi} \\ \boldsymbol{\omega}_v^{vi}\end{bmatrix}$, consisting of the linear and angular velocities
  - $\dot{\boldsymbol{T}}_{vi} = \begin{bmatrix}\boldsymbol{\omega}_v^{vi\times} & -\boldsymbol{\nu}_v^{vi} \\ \boldsymbol{0}^T & 0\end{bmatrix}\boldsymbol{T}_{vi}$

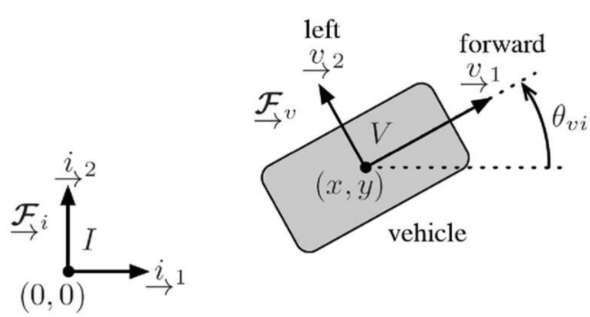

Figure 1: Illustration of the unicycle model.

- Example: we can derive the unicycle model by considering a 2D robot with position (x, y) and angle θ_{vi}, where axis z comes out of the plane

- $\boldsymbol{r}_i^{vi} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$, $\boldsymbol{C}_{vi} = \boldsymbol{C}_3(\theta_{vi}) = \begin{bmatrix} \cos\theta_{vi} & \sin\theta_{vi} \\ 0 & -\sin\theta_{vi} \end{bmatrix} \cos\theta_{vi} 0001$
    * Note the notation for elementary rotations is different; $\boldsymbol{C}_3(\theta)$ does not denote a rotational transformation by θ, rather it denotes the transformation from the current frame to a frame obtained by rotating by θ, so the formula for the rotation matrices is transposed compared to the usual ones
- For the pose we want $\boldsymbol{T}_{iv} = \begin{bmatrix} \boldsymbol{C}_{iv} & \boldsymbol{r}_i^{vi} \\ \boldsymbol{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_{vi} & -\sin\theta_{vi} & 0 & x \\ \sin\theta_{vi} & \cos\theta_{vi} & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
    * Note this is not $\boldsymbol{T}_{vi}$, so we use $\boldsymbol{C}_{iv}$ and not $\boldsymbol{C}_{vi}$!
    * Usually the third row and column is omitted for 2D
- For kinematics we constrain $\boldsymbol{\varpi}_v^{vi} = \begin{bmatrix} v & 0 & 0 & 0 & 0 & \omega \end{bmatrix}^T$, i.e. only forward movement and rotation along z
- Using the relation between $\dot{\boldsymbol{T}}_{iv}$ and $\boldsymbol{\varpi}_v^{vi}$ we get $\begin{cases} \dot{x} = v\cos\theta \\ \dot{y} = v\sin\theta \\ \dot{\theta} = \omega \end{cases}$