Tutorial 8, Mar 9, 2026

- Consider $x(k+1) = Ax + Bu$, and suppose there exists K such that $A + BK$ is Schur stable; we want to stabilize a nonzero equilibrium, $\bar{x}$, and suppose there exists $\bar{u}$ such that $\bar{x} = A\bar{x} + B\bar{u}$
  - Let $z(k) = x(k) - \bar{x}$
  - The error dynamics are $$\begin{aligned} z(k+1) &= x(k+1) - \bar{x} \\ &= Ax(k) + Bu(k) - A\bar{x} - B\bar{u} \\ &= A(x(k) - \bar{x}) + B(u(k) - \bar{u}) \\ &= Az(k) + B(u(k) - \bar{u}) \end{aligned}$$
  - We need to choose u such that A is stabilized and $\bar{u}$ is cancelled, so this suggests we choose $u(k) = Kz(k) + \bar{u} = K(x(k) - \bar{x}) + \bar{u}$
  - Now $z(k+1) = Az(k) + B(Kz(k) + \bar{u} - \bar{u}) = (A + BK)z(k)$
  - Since $A + BK$ is Schur stable, $x(k) \to \bar{x}$
- Consider $y(k+1) = ay(k) + bu(k)$ where a and b are unknown, $b \neq 0$ with sign known, $y(k)$ is measured, and we want to track $y_m(k+1) = a_m y_m(k) + b_m r(k)$ where $|a_m| < 1, b_m \neq 0$ and $r(k)$ is a bounded reference input
  - Assume a, b are known, find (θ_1, θ_2) so that the control law $u(k) = \theta_1 y(k) + \theta_2 r(k)$ achieves a perfect model matching, i.e. $y(k) = y_m(k)$ for all $k \geq 0$
    - $$\begin{aligned} y(k+1) &= ay(k) + bu(k) \\ &= ay(k) + b(\theta_1 y(k) + \theta_2 r(k) \\ &= (a + b\theta_1)y(k) + b\theta_2 r(k) \end{aligned}$$
    - We want this to equal $y_m(k+1) = a_m y_m(k) + b_m r(k)$
    - Therefore we need $a_m = a + b\theta_1, b_m = b\theta_2$, which are the matching conditions
  - Now assume a, b are unknown, and let $e(k) = y(k) - y_m(k)$; derive an error model in terms of $\tilde{\theta}(k) = \hat{\theta}(k) - \theta$
    - $$\begin{aligned} e(k+1) &= y(k+1) - y_m(k+1) \\ &= ay(k) + bu(k) - a_m y(k) - b_m r(k) \\ &= (a_m - b\theta_1)y(k) + b(\hat{\theta}_1 y(k) + \hat{\theta}_2 r(k)) - a_m y(k) - b_m r(k) && \text{Matching conditions} \\ &= a_m(y(k) - y_m(k)) + b(\hat{\theta}_1 y(k) + \hat{\theta}_2 r(k)) - b\theta_1 y(k) - b_m r(k) \\ &= a_m e(k) + b(\hat{\theta}_1 - \theta_1)y(k) + b(\hat{\theta}_2 - \theta_2)r(k) \\ &= a_m e(k) + b\tilde{\theta}^T(k)w(k) && w(k) = \begin{bmatrix} y(k) \\ r(k) \end{bmatrix} \end{aligned}$$
  - Propose an adaptive controller to make $e(k) \to 0$
    - Let $H(z) = \dfrac{1}{z - a_m}$
    - $e(k) = bH(z)\left[\tilde{\theta}^T w(k)\right]$
      - We isolated b since it's unknown
    - Define the augmented error and regressor as usual to get $e_a(k) = b\tilde{\theta}^T w_a(k)$
    - $\hat{\theta}(k+1) = \hat{\theta}(k) - \gamma(k) b e_a(k) w_a(k)$, but we don't have b
    - However, we can combine $b\bar{\gamma}$ into a single constant: $\hat{\theta}(k+1) = \hat{\theta}(k) - \gamma'(k)\,\text{sign}(b) e_a(k) w_a(k)$ where $\gamma'(k) = \dfrac{\bar{\gamma} b}{1 + \|w_a(k)\|}$, and just choose $\bar{\gamma} b \in (0, 2)$
  - The sign of b must be known sine otherwise we cannot make sure the gradient law is stable
- Consider the system $x(k+1) = Ax(k), y(k) = Cx(k)$ where $x(k) \in \mathbb{R}^2$
  - Is it possible for this system to produce an output $y(k) = k^2$?
    - We can view states as degrees of freedom, allowing us to build more complex functions with higher order systems
    - If $x(k) \in \mathbb{R}$ we can only produce constant functions; with $x(k) \in \mathbb{R}^2$ we can produce ramp functions and so on
    - To find the system we write $y(k)$ in terms of previous $y(k)$, e.g. for $y(k) = k$
      - $y(k+1) = k + 1 = y(k) + 1$

- $y(k+2) = y(k+1) + 1 = y(k+1) + (y(k+1) - y(k)) = 2y(k+1) - y(k)$
- Using this difference equation we can find a state space realization

* Now we try to do it for $y(k) = k^2$:
  - $y(k+1) = k^2 + 2k + 1 = y(k) + 2k + 1$
  - $y(k+2) = k^2 + 4k + 4$

$$\begin{aligned} &= y(k) + 2(2k+1) + 2 \\ &= y(k) + 2(y(k+1) - y(k)) + 2 \\ &= -y(k) + 2y(k+1) + 2 \end{aligned}$$

  - $y(k+3) = k^2 + 6k + 9$

$$\begin{aligned} &= y(k) + 3(2k+1) + 6 \\ &= y(k) + 3(y(k+1) - y(k)) + 3(y(k+2) + y(k) - 2y(k+1)) \end{aligned}$$