Tutorial 6, Feb 23, 2026

- Write a state-space model for the filtered regressor $w_f(k) = H(z)I[w(k)]$ where $H(z) = \frac{1}{z^2 - z + \frac{1}{4}}, w(k) = \begin{bmatrix} \sin k \\ 1 \end{bmatrix}$
  - First convert $H(z)$ to state space
    * $\frac{Y(z)}{U(z)} = \frac{1}{z^2 - z + \frac{1}{4}}$
    * Inverse Z transform: $y(k+2) - y(k+1) + \frac{1}{4}y(k) = u(k)$
    * Let $x_1(k) = y(k), x_2(k) = y(k+1)$
    * $\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} x_2(k) \\ x_2(k) - \frac{1}{4}x_1(k) + u(k) \end{bmatrix}$
  - Filter each component of the regressor separately
    * $x_1(k+1) = x_2(k)$
    * $x_2(k+1) = x_2(k) - \frac{1}{4}x_1(k) + w_1(k)$
    * $w_{f_1}(k) = x_1(k)$
    * $x_3(k+1) = x_4(k)$
    * $x_4(k+1) = x_4(k) - \frac{1}{4}x_3(k) + w_2(k)$
    * $w_{f_2}(k) = x_3(k)$
  - Combined system: $\begin{cases} x(k+1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{4} & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1(k) \\ w_2(k) \end{bmatrix} \\ w_f(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(k) \end{cases}$
- Determine if the equilibrium of the previous system is stable, AS, or ES using a Lyapunov argument
  - Since we have 2 decoupled systems that are identical, we can simply prove stability for one of the systems
  - $A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4} & 1 \end{bmatrix}$
  - Set $Q = I$, solve for P such that $A^T PA - P = -Q$
  - $\begin{bmatrix} 0 & -\frac{1}{4} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -\frac{1}{4} & 1 \end{bmatrix} - \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
  - After getting a solution, we use the Lyapunov function $V(x) = x^T Px$, with $\Delta V(x(k)) = -x^T Qx$ to prove exponential stability
- Consider the discrete-time system $x(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -1 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k), x(0) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and suppose the state is not measurable; find a sequence of inputs $u(k)$ such that $x(5) = 0$
  - Since we have $x(5) = 0$, we want to do deadbeat control
  - (A, B) is controllable since we have controllable canonical form
  - Let $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$ and find K such that $\sigma(A + BK) = \{0, 0, 0\}$
    * $A+BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3+k_1 & -1+k_2 & 1+k_3 \end{bmatrix} \implies \Delta_{A+BK}(s) = s^3 - (k_3+1)s^2 - (k_2-1)s - (k_1+3)$
    * To get the correct spectrum, take $K = \begin{bmatrix} -3 & 1 & -1 \end{bmatrix}$ and the controller $u(k) = Kx(k) = -3x_1(k) + x_2(k) - x_3(k)$

– $x(k)$ is not measurable, but we know the initial conditions and the system parameters, so we can simulate the system as $x(k+1) = (A + BK)x(k)$ and get $u(k) = Kx(k)$ for each step