# Tutorial 5, Feb 9, 2026

- Consider a system $\begin{cases} x(k+1) = \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} x(k) \\ y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \end{cases}$
  1. Given $y(k) = \cos(\omega k)$, develop an adaptive law that asymptotically recovers an initial condition which generates $y(k)$
     - In this case $\psi = x(0)$
     - $y(k) = CA^k x(0) = \begin{bmatrix} \cos(\omega k) & -\sin(\omega k) \end{bmatrix} \psi$
       * $A$ is a rotation matrix, which means $A^k = \begin{bmatrix} \cos(\omega k) & -\sin(\omega k) \\ \sin(\omega k) & \cos(\omega k) \end{bmatrix}$
     - We have $y(k) = \psi^T w(k)$ where $w(k) = \begin{bmatrix} \cos(\omega k) \\ -\sin(\omega k) \end{bmatrix}$, which suggests $w(k)$ being the regressor
     - Let $\hat{y} = \hat{\psi}^T w(k), e(k) = \hat{y}(k) - y(k) = (\hat{\psi} - \psi)^T w(k) = \tilde{\psi}^T(k) w(k)$, giving a static error model
       * What we did here was to construct the signal $y(k)$ using the parameter we're trying to estimate, and then derive an error model
     - Use the adaptive law $\hat{\psi}(k+1) = \hat{\psi}(k) - \frac{\bar{\gamma}}{1 + \|w(k)\|^2} e(k) w(k), \bar{\gamma} \in (0, 2)$
  2. If $y(k) = \sin(2\omega k)$, can we still obtain the correct $\psi$ estimate using the same adaptive law?
     - Notice that $\sin(2\omega k)$ cannot be written as a linear combination of $\cos(\omega k)$ and $-\sin(\omega k)$, so there is no such $\psi$ such that $y(k) = \psi^T w(k)$ and $e(k)$ doesn't go to zero
  3. Suppose $y_{\text{meas}}(k) = y(k) + \eta(k)$ where $\eta(k)$ is some scalar high frequency noise; to attenuate the noise, we use a stable filter $H(z)$ to produce $y_{\text{filt}}(k) = H(z)\left[y_{\text{meas}}(k)\right]$; assuming $H(z)\left[\eta(k)\right] = 0$, develop an adaptation law that uses $y_{\text{filt}}(k)$ to recover initial conditions which generates $y(k)$
     - $$\begin{aligned} y_{\text{filt}}(k) &= H(z)\left[y_{\text{meas}}(k)\right] \\ &= H(z)\left[y(k) + \eta(k)\right] \\ &= H(z)\left[y(k)\right] + H(z)\left[\eta(k)\right] \qquad \text{by linearity, since } H(z) \text{ is stable} \\ &= H(z)\left[\psi^T w(k)\right] \end{aligned}$$
     - Define $\hat{y}_{\text{filt}}(k) = H(z)\left[\hat{\psi}^T(k) w(k)\right]$
     - $$\begin{aligned} e(k) &= \hat{y}_{\text{filt}}(k) - y_{\text{filt}}(k) \\ &= H(z)\left[\hat{\psi}^T(k) w(k)\right] - H(z)\left[\psi^T w(k)\right] \\ &= H(z)\left[\hat{\psi}^T(k) w(k) - \psi^T w(k)\right] \\ &= H(z)\left[\tilde{\psi}^T(k) w(k)\right] \end{aligned}$$
       * This is a dynamic error model, since $e(k)$ is is the output of a stable dynamical system
     - $$\begin{aligned} e(k) &= H(z)\left[\hat{\psi}^T(k) w(k)\right] - H(z)\left[\psi^T w(k)\right] \\ &= H(z)\left[\hat{\psi}^T(k) w(k)\right] - \psi^T H(z) I\left[w(k)\right] \end{aligned}$$
     - Let $$\begin{aligned} e_a(k) &= e(k) - H(z)\left[\hat{\psi}^T(k) w(k)\right] + \hat{\psi}^T(k) H(z) I\left[w(k)\right] \\ &= \hat{\psi}^T(k) H(z) I\left[w(k)\right] - \psi^T H(z) I\left[w(k)\right] \\ &= \tilde{\psi}^T(k) H(z) I\left[w(k)\right] \\ &= \tilde{\psi}^T(k) w_a(k) \end{aligned}$$
     - Now we can use the adaptive law $\hat{\psi}(k+1) = \hat{\psi}(k) - \frac{\bar{\gamma}}{1 + \|w_a(k)\|^2} e_a(k) w_a(k), \bar{\gamma} \in (0, 2)$
       * Note we have $e_a(k) = \hat{\psi}^T(k) w_a(k) - y_{\text{filt}}(k)$ since the first piece is $\hat{\psi}^T(k) H(z) I\left[w(k)\right]$ and the second piece is $\psi^T H(z) I\left[w(k)\right]$ after applying the swapping lemma to the original definition of $y_{\text{filt}}$