Tutorial 12, Mar 30, 2026

- Consider the system: $x(k+1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$

  $y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$

  - Design a regulator to make $y(k)$ track $r(k) = \bar{r} \cdot 1(k)$ (i.e. a step input)
    * First, we build an exosystem for the disturbance and reference
      - Since there is no disturbance and a constant reference, we simply have $S = 1$ and $w(0) = \bar{r}$
      - $e(k) = Cx(k) + Dw(k) = y(k) - r(k) = y(k) - w(k)$ and therefore $D = -1$
    * Design the feedback gain
      - $A + BK = \begin{bmatrix} k_1 & k_2 + 1 \\ k_1 - 1 & k_2 \end{bmatrix}$
      - $\det(zI - (A + BK)) = z^2 + (-k_1 - k_2)z + 1 + k_2 - k_1$
      - We choose $\Delta_d(z) = z^2 + \frac{1}{4}$ (arbitrary as long as $|\lambda| < 1$)
      - This gives us $K = \begin{bmatrix} \frac{3}{8} & -\frac{3}{8} \end{bmatrix}$
    * Solve the regulator equations
      - $\begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Gamma$
      - $\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} - 1 = 0$
      - We get $\Pi = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \Gamma = 1$
    * The state feedback solution is $u(k) = w(k) + \begin{bmatrix} \frac{3}{8} & -\frac{3}{8} \end{bmatrix} \left(x(k) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(k) \right)$
    * Since $x(k)$ and $w(k)$ are not directly measurable we design the observers
      - We want $\bar{A} - L\bar{C}$ to be Schur stable, where $\bar{A} = \begin{bmatrix} A & E \\ 0 & S \end{bmatrix}, \bar{C} = \begin{bmatrix} C & D \end{bmatrix}$
      - $\bar{A} - L\bar{C} = \begin{bmatrix} -l_1 & 1 & l_1 \\ -1 - l_2 & 0 & l_2 \\ -l_3 & 0 & 1 + l_3 \end{bmatrix}$
      - $\det(zI - (\bar{A} - L\bar{C})) = z^2 + (l_1 - 1 - l_3)z^2 + (1 + l_2 - l_1)z - 1 - l_2 - l_3$
      - Choose to have $\Delta_d(z) = z^3 \implies L = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$
        - Note this is a deadbeat observer
      - $\hat{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \hat{x}(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) + \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} (e(k) - \hat{e}(k))$
      - $\hat{w}(k+1) = \hat{w}(k) - \frac{1}{2}(e(k) - \hat{e}(k))$
      - $\hat{e}(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{x}(k) - \hat{w}(k)$