Tutorial 1, Jan 12, 2026

Discrete-Time Models

- Consider the system $y(k+2) - 6y(k+1) + 8y(k) = u(k)$
  - Find its transfer function $G(z) = \dfrac{Y(z)}{U(z)}$
    - $$\begin{aligned} & z^2Y(z) - 6zY(z) + 8Y(z) = U(z) \\ \implies & Y(z)(z^2 - 6z + 8) = U(z) \\ \implies & \frac{Y(z)}{U(z)} = \frac{1}{z^2 - 6z + 8} \end{aligned}$$
  - Find its state-space realization
    - Recall that the number of required states is equal to the largest forward shift in the difference equation, so we need two states
    - Let $x_1(k) = y(k), x_2(k) = y(k+1) \implies x(k) = \begin{bmatrix} y(k) \\ y(k+1) \end{bmatrix}$
    - $x(k+1) = \begin{bmatrix} y(k+1) \\ y(k+2) \end{bmatrix} = \begin{bmatrix} x_2(k) \\ u(k) - 8x_1(k) + 6x_2(k) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -8 & 6 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$
    - $y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$
    - This system is now in controllable canonical form
- Consider the system $y(k+2) - 4y(k+1) + 3y(k) = u(k+1) + 2u(k)$
  - Find its transfer function
    - $z^2Y(z) - 4zY(z) + 3Y(z) = zU(z) + 2U(z) \implies \dfrac{Y(z)}{U(z)} = \dfrac{z+2}{z^2 - 4z + 3}$
  - Find its state-space realization
    - Since there is a forward shift on the input, we need to break up the system into a state equation and a measurement equation, which is easiest with the transfer function
    - Let $V(z) = \dfrac{1}{z^2 - 4z + 3}U(z), Y(z) = (z+2)V(z)$
    - Let $x = \begin{bmatrix} v(k) \\ v(k+1) \end{bmatrix}$
    - State equation:
      - Inverse Z-transform: $v(k+2) - 4v(k+1) + 3v(k) = u(k)$
      - $x(k+1) = \begin{bmatrix} x_2(k) \\ u(k) - 3x_1(k) + 4x_2(k) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$
    - Measurement equation: $y(k) = v(k+1) + 2v(k) = x_2(k) + 2x_1(k) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$
- For a system in controllable canonical form, $A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$, it has characteristic polynomial $s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$