Lecture 6, Jan 16, 2026

Stability of Discrete-Time Systems

- Consider a general open-loop nonlinear system $x(k+1) = f'(x(k), u(k))$ and its closed-loop system, $x(k+1) = f(k, x(k))$, where u no longer appears since we have designed some control law
  - Note that since we have k appearing explicitly in f, our systems are not time-invariant; this is necessary when we talk about adaptive control

Definition

A constant vector $\bar{x} \in \mathbb{R}^n$ is an *equilibrium* of the closed-loop system $x(k+1) = f(k, x(k))$ if $\bar{x} = f(k, \bar{x})$.

- Notice that whereas in continuous time the equilibria are points where f is zero, in discrete time equilibria are fixed points of f
- To make initial conditions explicit, denote $x(k) \equiv x(k; k_0, x_0)$ which means $x(k_0) = x_0$

Definition

Let $\bar{x} \in \mathbb{R}^n$ be an equilibrium of the system $x(k+1) = f(k, x(k))$, then

1. $\bar{x}$ is *stable* if $\forall k_0 \geq 0, \varepsilon > 0, \exists \delta(\varepsilon, k_0) > 0$ s.t. $\|x_0 - \bar{x}\| < \delta \implies \|x(k; k_0, x_0) - \bar{x}\| < \varepsilon, \forall k \geq k_0$.
2. $\bar{x}$ is *asymptotically stable* if it's stable and $\exists \delta(k_0) > 0$ s.t. $\|x_0 - \bar{x}\| < \delta \implies \lim_{k \to \infty} x(k; k_0, x_0) = \bar{x}$ (attractivity condition).
3. $\bar{x}$ is *uniformly asymptotically stable* if it's asymptotically stable and δ in the previous definitions are independent of k_0.
4. $\bar{x}$ is *globally asymptotically stable* if it is asymptotically stable (or *globally uniformly asymptotically stable* if it is also uniformly asymptotically stable), and $\delta(k_0)$ can be arbitrary large, i.e. all initial conditions converge to $\bar{x}$.

- The definition of stability is analogous to the continuous time definition; it requires that for any positive ε, we can find δ such that starting within δ of the equilibrium guarantees that the solution never goes outside ε of the equilibrium
- Similarly for asymptotic stability, like in the continuous case, we require that solutions near the equilibrium converge to the equilibrium
- Uniform asymptotic stability is important for reasons of robustness
  - Note that this is only an issue for time-dependent systems; for time-invariant systems we never have this k_0 dependence, but adaptive control is time-dependent
- In general GUAS is the best outcome