Lecture 5

Qualitative Behaviour of Solutions

- Consider a system $x(k+1) = Ax(k) + Bu(k)$ with response $x(k) = A^k x(0) + \sum_{i=0}^{k-1} A^{k-1-i} Bu(i)$; what can we conclude qualitatively about the solution without explicitly computing it?
- Let the system have output $y(k) = Cx(k)$ and transfer function $G(z) = C(zI - A)^{-1}B = k\frac{(z-q_1)(z-q_2)\cdots}{(z-p_1)(z-p_2)\cdots}$
- To get $Y(k)$ we use partial fractions as with Laplace transforms, yielding $\frac{c_1}{z-p_1} + \frac{c_2}{z-p_2} + \cdots$
- What is the behaviour of a typical pole?
- For real distinct poles, $\mathcal{Z}^{-1}\left\{\frac{z}{z-p}\right\} = p^k$
  - Clearly, if $p = 1$ then we get a constant response, or if $p = -1$ we get an alternating response that has constant magnitude
  - For $p \in (0, 1)$ we get a decaying envelope that does not change sign; with the decay faster the closer p is to zero
  - For $p = 0$ we get a constant zero (but we need to watch out since now we have z/z)
  - For $p \in (-1, 0)$ we get a decaying envelope with alternating signs, again with decay faster the closer p is to zero
  - In summary:
    - $p < 0 \to$ solution alternates signs
    - $|p| < 1 \to$ solution decays
    - $|p| > 1 \to$ solution blows up
    - $p = \pm 1 \to$ steady-state response
- For complex conjugate poles, $\mathcal{Z}^{-1}\left\{\frac{z^2}{(z-re^{j\omega})(z-re^{-j\omega})}\right\} = \frac{1}{\sin\omega} r^k \sin(k\omega + \phi)$
  - This contains a constant factor, an exponential envelope r^k and an oscillation at frequency ω
  - Larger values of r decay slower, until $r = 1$ which is a steady-state oscillation, then for $r > 1$ the solution blows up
  - The frequency of oscillation gets faster with increasing ω, as we increase the angle of the pole
    - However since the angles live on a circle, if we have $\omega > \pi$ (i.e. flipping beyond the negative real line), the effect is the same as reducing the frequency
    - This is because $\omega = \pi$ represents the Nyquist frequency, and any higher frequency contents will be aliased into lower frequencies in the output, so we have a fundamental limit based on the sampling rate
  - In summary:
    - Poles on the same circle have the same exponential envelope
    - Poles with the same angle have the same oscillation frequency