Lecture 4, Jan 13, 2026

Computing the Matrix Power

- Given $A \in \mathbb{R}^{n \times n}$, we want to find a closed-form expression for A^k so we can compute the time response
- Let $\sigma(A) = \{ \lambda_1, \dots, \lambda_n \}, \lambda_i \in \mathbb{C}$ denote the spectrum of A, i.e. the set of eigenvalues of A
- Assume that A is diagonalizable (recall that this is equivalent to A having n linearly independent eigenvectors), then we can compute A^k as $A^k = P\Lambda^k P^{-1}$ where P is the matrix of eigenvectors which diagonalizes A
  - In general for nondiagonalizable A this will be replaced with the Jordan form
- Example: $A = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}$
  - This has characteristic polynomial $s^2 + 2s - 3 = (s+3)(s-1)$ so $\sigma(A) = \{ 1, -3 \}$
  - Substitute λ in $\begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \lambda \begin{bmatrix} v_{11} \\ v_{22} \end{bmatrix}$ and solve for eigenvectors to get $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$
  - $P = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix} \implies A^k = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & (-3)^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}^{-1}$
- Note that A^k can also be computed using the Z-transform (similarly to how e^{At} can be computed with the Laplace transform)
  - Starting with $x(k+1) = Ax(k)$, which we know the solution to be $x(k) = A^k x(0)$
  - Apply Z-transforms to get
$$\begin{aligned} & zX(z) - zx(0) = AX(z) \\ \implies & X(z) = (zI - A)^{-1} z x(0) \\ \implies & x(k) = \mathcal{Z}^{-1}\left\{ (zI - A)^{-1} z \right\} x(0) \end{aligned}$$
    - * Note we don't assume zero initial conditions here since we're not trying to derive a transfer function
  - By existence and uniqueness of solutions, we conclude that $A^k = \mathcal{Z}^{-1}\left\{ (zI - A)^{-1} z \right\} x(0)$
    - * Note that this inverse Z-transform can be computed componentwise using the residue theorem (not covered in this course)

Poles and Eigenvalues

- The qualitative behaviour of solutions can be inferred by looking at either the eigenvalues of A or the poles of the transfer function
- What is the relationship between poles and eigenvalues of A?
- Consider the output equation $y(k) = Cx(k)$ with transfer function (SISO) $\frac{Y(z)}{U(z)} = C(zI - A)^{-1}B$ as we've derived last time
  - Expanding the inverse we get $C\left(\frac{\operatorname{adj}(zI - A)}{\det(zI - A)} \right) B = \frac{N'(z)}{D'(z)}$
    - * Note that the middle term is a scalar, so we can conclude that $D'(z)$ is the characteristic polynomial of A
  - To get an actual transfer function we need to cancel roots so that the numerator and denominator are coprime, to get $\frac{N(z)}{D(z)}$
  - This means that all the poles (roots of $D(z)$) are eigenvalues of A, but not the other way around due to pole-zero cancellations – some information is lost when we convert from state space to transfer function
    - * A system that has a stable transfer function might not necessarily be stable in all its states