Lecture 33, Mar 31, 2026

Review – Part 1

Modelling of Discrete Time Systems

- Important concepts: difference equations, transfer functions, state-space models; coordinate transformations $z(k) = Px(k)$
  - Example: Given $y(k+2) - y(k-1) = y(k)$ with $y(2) = 2, y(3) = 0, y(-2) = 1$; find a state space representation and the initial conditions of the state
    - Put into standard form: $y(k) = y(k-3) + y(k-2)$, therefore we have 3 states
    - $x(k) = \begin{bmatrix} y(k) \\ y(k+1) \\ y(k+2) \end{bmatrix}$
    - State space model: $x(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} x(k)$
    - For initial conditions, we need $y(0), y(1), y(2)$ for $x(0)$, but we are given y at different times, so we must use the difference equation to relate what we're given to what we need and solve for it
      - $y(0) = y(-3) + y(-2)$
      - $y(1) = y(-2) + y(-1) \implies y(1) = 2$
      - $y(2) = y(-1) + y(0) \implies y(-1) = 1$ from $y(0)$
      - $y(3) = y(0) + y(-2) \implies y(0) = -1$ from what we're given
- Time response (computing A^k): stability definitions, stability of LTI systems; types of stability (asymptotic, exponential, BIBO, etc)
  - Example: Consider an LTI system $x(k+1) = Ax(k)$, where $A \in \mathbb{R}^{2\times 2}$ is diagonalizable
    - Given $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we get $x(k) = \begin{bmatrix} 0.5^k \\ 0.5^k - 1 \end{bmatrix}$. Is $x = 0$ stable, asymptotically stable, or not enough information?
      - Since we see two different frequencies (the 0.5^k and the constant -1), the system eigenvalues must be at 0.5 and 1
      - $x(k) = P\Lambda^k P^{-1} x(0) = P \begin{bmatrix} 0.5^k & 0 \\ 0 & 1 \end{bmatrix} P^{-1} x(0)$
      - Since the system can only have 2 eigenvalues since it's second order, these must be all the eigenvalues; so the system is stable but not asymptotically stable
    - Given $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we get $x(k) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; find A
      - Let $x^1(k)$ denote the response from the first experiment and $x^2(k)$ the response from the second
      - $\begin{bmatrix} x^1(k) & x^2(k) \end{bmatrix} = A^k \begin{bmatrix} x^1(0) & x^2(0) \end{bmatrix} \implies \begin{bmatrix} 0.5^k & 0 \\ 0.5^k - 1 & 1 \end{bmatrix} = P \begin{bmatrix} 0.5^k & 0 \\ 0 & 1 \end{bmatrix} P^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
      - Therefore we must find P to get A
      - We know $A^k v_1 = \lambda_1^k v_1 \implies \begin{bmatrix} 0.5^k & 0 \\ 0.5^k - 1 & 1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0.5^k v_{11} \\ 0.5^k v_{12} \end{bmatrix}$
        - Therefore $v_{11} = 1$ and $v_{12} = 1$
      - We can repeat for λ_2: $\begin{bmatrix} 0.5^k & 0 \\ 0.5^k - 1 & 1 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix}$ so $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
      - Therefore $P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 0.5 & 0 \\ -0.5 & 1 \end{bmatrix}$
    - Given $x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, compute $x(k)$
      - We can use linearity; by superposition $x(k) = x^1(k) + 3x^2(k) = \begin{bmatrix} 0.5^k \\ 0.5^k + 2 \end{bmatrix}$

- Consider the system: $x(k+1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k), y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(k)$
  - * Prove $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a stable equilibrium using the definition
    - Assume $u(k) = 0$, then $x_1(k+1) = x_1(k), x_2(k+1) = 0$
    - Therefore $x_1(k) = x_1(0)$ and $x_2(k) = \begin{cases} x_2(0) & k = 0 \\ 0 & k > 0 \end{cases}$
    - Now let $\varepsilon > 0$ be arbitrary, and choose $\delta = \varepsilon$, then $\|x(0)\| < \delta \implies \|x(k)\| \leq \|x(0)\| < \delta = \varepsilon$, so by definition the equilibrium is stable
  - * Suppose $u(k) = 0, \forall k \geq 0$; find $x(0)$ such that $x(k) \to 0$ as $x \to \infty$
    - As we have found in the last question, we just need $x(0) = \begin{bmatrix} 0 \\ a \end{bmatrix}$ where a is arbitrary