Lecture 32, Mar 27, 2026

Adaptive Regulation – Summary

- Now that we have all the pieces required, we can put them together to perform adaptive regulation
- Consider the system $x(k+1) = Ax(k) + Bu(k) + E\xi(k)$ where:
$$\begin{aligned}\xi(k+1) &= S\xi(k)\\ e(k) &= Cx(k) + D\xi(k)\end{aligned}$$
  - S, E, D are unknown
  - A, B, C are known
  - $e(k)$ is measurable

Theorem

For the exosystem
$$\begin{aligned}\xi(k+1) &= S\xi(k)\\ d(k) &= \Gamma\xi(k)\end{aligned}$$
where $d(k) \in \mathbb{R}$, assuming (Γ, S) is observable, (F, G) controllable, $\sigma(F) \cap \sigma(S) = \varnothing$, and F is Schur stable, then there exists a nonsingular $M \in \mathbb{R}^{q\times q}$ which solves the Sylvester equation $MS = FM + G\Gamma$.

- With the coordinate transformation $w(k) = M\xi(k)$, we get the exosystem $w(k+1) = Fw(k) + Gd(k)$
$$d(k) = \psi^T w(k)$$
  - Note that since $w(k+1) = (F + G\psi^T)w(k)$, we have $\sigma(S) = \sigma(F + G\psi^T)$
- Assume that a solution to the regulator equations exists: $\Pi S = A\Pi + B\Gamma + E$
$$0 = C\Pi + D$$
  1. (A, B) is controllable
  2. (C, A) is observable
  3. $\sigma(S)$ are *simple* (i.e. algebraic multiplicity 1) and on the unit circle in $\mathbb{C}$
  4. *Nonresonance condition*: $\det \begin{bmatrix} A - \lambda I & B \\ C & 0 \end{bmatrix} \neq 0, \forall \lambda \in \sigma(S)$
     - We are checking if any of the frequencies in the exosystem will be cancelled by the plant
     - This is similar to making sure there are no pole zero cancellations between the exosystem and plant
- Let the error $z(k) = x(k) - \Pi\xi(k)$, which has dynamics $z(k+1) = Az(k) + Bu(k) - B\Gamma\xi(k)$
$$\begin{aligned} &= Az(k) + Bu(k) - B\psi^T w(k)\\ e(k) &= Cz(k)\end{aligned}$$
- We use the controller $u(k) = u_s(k) + u_{im}(k)$
  - $u_s(k)$ is the stabilizer, and we expect $u_s(k) \to 0$
  - $u_{im}(k)$ contains the internal model, and we expect this to be a steady state
- We use the disturbance observer $\hat{z}_d(k+1) = A\hat{z}_d(k) + Bu(k) + L_d(e(k) - C\hat{z}_d(k))$ where we choose L_d such that $A - L_dC$ is Schur stable
  - Let the estimation error $\tilde{z}_d(k) = \hat{z}_d(k) - z(k) \implies \tilde{z}_d(k+1) = A_d\tilde{z}_d(k) + Bd(k)$ where $A_d = A - L_dC$
  - Define the filtered disturbance $d_f(k) = C\tilde{z}_d(k) = C\hat{z}_d(k) - e(k)$, which can be interpreted as $d(k)$ filtered through the LTI system $\tilde{z}_d(k+1) = A_d\tilde{z}_d(k) + Bd(k)$
$$d_f(k) = C\tilde{z}_d(k)$$
    - Note that $d_f(k)$ is measurable
    - Note that technically, depending on the initial conditions, $d_f(k)$ will have transients, but $d(k)$ is steady-state; therefore we rely on the lemma below

Theorem

For the exosystem

$$\begin{aligned} w(k+1) &= Fw(k) + Gd(k) \\ d(k) &= \psi^T w(k) \end{aligned}$$

Let $A_d = A - L_d C$ be Schur stable, then there exists initial conditions $z_d(0) \in \mathbb{R}^n$ such that the steady-state $d_{f,ss}(k)$ obtained by

$$\begin{aligned} z_d(k+1) &= A_d z_d(k) + Bd(k) \\ d_{f,ss}(k) &= Cz_d(k) \end{aligned}$$

can be generated by a model

$$\begin{aligned} w_f(k+1) &= Fw_f(k) + Gd_{f,ss}(k) \\ d_{f,ss}(k) &= \psi^T w_f(k) \end{aligned}$$

and there exists a nonsingular $H_f \in \mathbb{R}^{q \times q}$ such that $w(k) = H_f w_f(k)$.

- Intuitively, the above is due to the fact that LTI systems don't change frequency content, so the filtered disturbance $d_f(k)$ should have the same frequencies as the unfiltered $d(k)$; therefore $d_f(k)$ can be generated with a system with the same F and G that generated $d(k)$, and the two systems differ only by a coordinate transformation
- To get $w_f(k)$, we build the observer $$\begin{aligned} \hat{w}_f(k+1) &= F\hat{w}_f(k) + Gd_f(k) \\ &= F\hat{w}_f(k) + G(C\hat{z}_d(k) - e(k)) \end{aligned}$$
  - This leads to exponential convergence of $w_f(k) \to w(k)$
- Now the disturbance becomes $$\begin{aligned} d(k) &= \Gamma\xi(k) \\ &= \psi^T w(k) \\ &= \psi^T H_f^{-1} w_f(k) \\ &= \psi_f^T w_f(k) \\ &= \psi_f^T \hat{w}_f(k) + \varepsilon(k) \qquad \text{where } \varepsilon(k) \to 0 \end{aligned}$$
  - Substitute into error dynamics: $$\begin{aligned} z(k+1) &= Az(k) + Bu(k) - B\psi^T w(k) \\ &= Az(k) + Bu(k) - B\psi_f^T w_f(k) \end{aligned}$$
    - * This suggests we choose $u_{im}(k) = \hat{\psi}_f^T(k)\hat{w}_f(k)$ where $\hat{\psi}_f$ is our parameter estimate, so we can cancel out the last term
- Now $$\begin{aligned} z(k+1) &= Az(k) + Bu_s(k) + B\tilde{\psi}_f^T(k)\hat{w}_f(k) + \varepsilon(k) \\ e(k) &= Cz(k) \end{aligned}$$
  - This is not quite the dynamic error model yet, due to the fact that A might not be Schur stable, and the additional $Bu_s(k)$
  - However if A is stable, then we may set $u_s(k) = 0$ and directly obtain the dynamic error model, and do the standard dynamic EM adaptation law with the regressor filtered through the $z(k)$ system
- If A is unstable, we use an observer to estimate $z(k)$ and use a state feedback $u_s(k) = K\hat{z}_s(k)$, where $A + BK$ is Schur stable
  - Observer: $\hat{z}_s(k+1) = A\hat{z}_s(k) + Bu_s(k) + L_s(e(k) - C\hat{z}_s(k))$
- We also need a new adaptation law for $\hat{\psi}_f(k)$, which can be derived using a swapping argument on the disturbance observer:

  - $$\begin{aligned} d_f(k) &= H_d(z)\left[d(k)\right] && \text{where } H_d(z) = C(zI - A_d)^{-1}B \\ &= H_d(z)\left[\psi^T(k)w(k)\right] \\ &= H_d(z)\left[\psi_f^T(k)w_f(k)\right] \\ &= \psi_f^T(k)H_d(z)I\left[w_f(k)\right] + \varepsilon(k) && \text{swapping lemma} \\ &= \psi_f^T(k)H_d(z)I\left[\hat{w}_f(k)\right] + \varepsilon(k) \end{aligned}$$
  - Define $\hat{d}_f(k) = \hat{\psi}_f^T(k)H_d(z)I\left[\hat{w}_f(k)\right]$, which is measurable
  - Let the augmented error $e_a(k) = \hat{d}_f(k) - d_f(k) = \tilde{\psi}_f^T\hat{w}_a(k)$ where $\hat{w}_a(k) = H_d(z)I\left[\hat{w}_f(k)\right]$
    - Recall that $d_f(k) = C\hat{z}_d(k) - e(k)$, so $e_a(k)$ is also measurable
    - This is now in the form of a static error model!
  - If A is Schur stable, we can still use this adaptation law, but in that case we can also directly use dynamic EM with $z(k)$
- The final control design:
  $$\begin{aligned} \hat{z}_d(k+1) &= A\hat{z}_d(k) + Bu(k) + L_d(e(k) - C\hat{z}_d(k)) \\ d_f(k) &= C\hat{z}_d(k) - e(k) \\ \hat{w}_f(k+1) &= F\hat{w}_f(k) + Gd_f(k) \\ u(k) &= u_s(k) + u_{im}(k) \\ u_{im}(k) &= \hat{\psi}_f^T(k)\hat{w}_f(k) \\ \hat{\psi}_f(k+1) &= \hat{\psi}_f(k) - \gamma(k)e_a(k)w_a(k) \\ \hat{w}_a(k) &= H_d(z)I\left[\hat{w}_f(k)\right] \\ e_a(k) &= \hat{\psi}_f^T(k)\hat{w}_a(k) - d_f(k) \\ \gamma(k) &= \frac{\bar{\gamma}}{1 + \|w_a(k)\|^2}, \bar{\gamma} \in (0, 2) \end{aligned}$$
  - Note, $H_d(z) = C(zI - A_d)^{-1}B$ is the LTI system that takes $d(k)$ to $d_f(k)$
  - If A is unstable:
    $$\begin{aligned} u_s(k) &= K\hat{z}_s(k) \\ \hat{z}_s(k+1) &= A\hat{z}_s(k) + Bu_s(k) + L_s(e(k) - C\hat{z}_s(k)) \end{aligned}$$
  - If A is stable: $u_s(k) = 0$
    - In this case we can also choose to make $\hat{w}_a(k), e_a(k)$ based on the dynamic error model with $z(k)$