Lecture 29, Mar 20, 2026

The Adaptive Regulator Problem

- Consider a model $x(k+1) = Ax(k) + Bu(k) + E\xi(k)$ where S is unknown (note w is renamed to ξ)

$$\xi(k+1) = S\xi(k)$$
$$e(k) = Cx(k) + D\xi(k)$$

  - In Lab 4, for the visuomotor adaptation problem, the disturbance enters the system via the error measurement only so $E = 0$
  - We don't know the frequency content of the possible disturbance, so S is unknown
  - To regulate this system we will need adaptive control, but first we need to work this into an error model
- We use the same idea from the normal regulator problem to transform: $\begin{cases} \Pi S = A\Pi + B\Gamma + E \\ 0 = C\Pi + D \end{cases}$
  - We can impose conditions to guarantee the existence of (Π, Γ), but we don't know what they are since we don't have S
  - Perform the same coordinate transform $z(k) = x(k) - \Pi\xi(k)$ (no longer measurable)
  - The resulting system is $z(k+1) = Az(k) + Bu(k) - B\Gamma\xi(k)$

$$e(k) = Cz(k)$$

- To make this into a dynamic error model, we have 3 issues: stabilization of the system, Γ is unknown, and $\xi(k)$ is unknown
- We have to figure out a suitable parametrization of the disturbance into a single parameter vector – known as the *Nikiforov canonical representation*
  - Let $d(k) = \Gamma\xi(k)$, then we have a system $\xi(k+1) = S\xi(k)$ (note that $d(k)$ is scalar)

$$d(k) = \Gamma\xi(k)$$

  - Pick (F, G) controllable and F Schur stable, then if (Γ, S) is observable and $\sigma(S) \cap \sigma(F) = \varnothing$, then the *Sylvester equation* $MS = FM + G\Gamma$ has a unique, nonsingular solution $M \in \mathbb{R}^{q\times q}$
    * Note we cannot actually compute M; we just know that it exists
    * As previously mentioned, if (Γ, S) is not observable then we can get rid of the extra states
    * Note that if F is Schur stable then it must have $|\lambda| < 1$, but if $\xi(k)$ contains only persistent signals then it must have $|\lambda| = 1$ (otherwise it would decay), so their spectra are naturally disjoint
  - Define a coordinate transformation $w(k) = M\xi(k)$, then

$$\begin{aligned} w(k+1) &= M\xi(k+1) \\ &= MS\xi(k) \\ &= (FM + G\Gamma)\xi(k) \\ &= Fw(k) + Gd(k) \end{aligned}$$

  - Now $d(k) = \Gamma\xi(k) = \Gamma M^{-1}w(k)$, so we can let $d(k) = \psi^T w(k)$
  - This gives us the new exosystem $w(k+1) = Fw(k) + Gd(k)$

$$d(k) = \psi^T w(k)$$

    * Note that if we substitute the second equation into the first, $w(k+1) = (F + G\psi^T)w(k)$, so ψ^T decides the frequency content of the disturbance
  - Now we substitute this back to get $z(k+1) = Az(k) + Bu(k) - Bd(k)$

$$e(k) = Cz(k)$$

- Let $u(k) = \hat{\psi}^T(k)w(k)$, then $z(k+1) = Az(k) + \tilde{\psi}^T(k)w(k)$ where $\tilde{\psi}(k) = \hat{\psi}(k) - \psi$, which is exactly the dynamic error model
- However, $w(k)$ is still not known, so we need to estimate it as well