Lecture 27, Mar 17, 2026

The Regulator Problem – Full Measurement

- To study this problem, we need to make an error model with states $z(k)$, such that $z(k) \to 0 \implies e(k) \to 0$
  - To do this, we try to find steady-state values $x_{ss}(k)$ and $u_{ss}(k)$, such that the error is zero when the system reaches these steady state values; then we can have $z(k) = x(k) - x_{ss}(k)$
  - The steady state is a solution, so $\begin{cases} x_{ss}(k+1) = Ax_{ss}(k) + Bu_{ss}(k) + Ew(k) \\ w(k+1) = Sw(k) \\ 0 = Cx_{ss}(k) + Dw(k) \end{cases}$
    * Note that we assume $w(k)$ contains only persistent (i.e. non-transient) exogenous signals
- Because LTI systems cannot change the frequency of the input, the frequency content of $x_{ss}(k)$ must come from $w(k)$ (this can be proven)
  - Therefore $\exists(\Pi, \Gamma)$ s.t. $x_{ss}(t) = \Pi w(k)$ and $u_{ss}(k) = \Gamma w(k)$
  - Substituting, we get $\begin{cases} \Pi S w(k) = (A\Pi + B\Gamma + E)w(k) \\ 0 = (C\Pi + D)w(k) \end{cases}$
  - Since $w(k)$ is arbitrary (as we have not fixed an exosystem), $\begin{cases} \Pi S = A\Pi + B\Gamma + E \\ 0 = C\Pi + D \end{cases}$
  - These are known as the *regulator equations*
- Now let $z(k) = x(k) - x_{ss}(k) = x(k) - \Pi w(k)$
  - $\begin{aligned} z(k+1) &= x(k+1) - \Pi w(k+1) \\ &= Ax(k) + Bu(k) + Ew(k) - \Pi S w(k) \\ &= Az(k) + A\Pi w(k) + Bu(k) + Ew(k) - \Pi S w(k) \\ &= Az(k) + Bu(k) + (A\Pi + E - \Pi S)w(k) \\ &= Az(k) + Bu(k) - B\Gamma w(k) \end{aligned}$
  - $\begin{aligned} e(k) &= Cx(k) + Dw(k) \\ &= C(z(k) + \Pi w(k)) + Dw(k) \\ &= Cz(k) + (C\Pi + D)w(k) \\ &= Cz(k) \end{aligned}$
  - $\Gamma w(k)$ is the *effective disturbance*
    * Notice that the channel that the effective disturbance enters through the same channels as $u(k)$, so now we can cancel it out
- Therefore we will use $u(k) = u_s(k) + u_{im}(k)$, where $u_s(k)$ gives us the desired steady-state, and $u_{im}(k)$ cancels the disturbance based on an internal model
  - $u_s(k) = Kz(k)$ such that $(A + BK)$ is Schur stable (pole placement)
    * If we had full measurements, we would have $u_s(k) = K(x(k) - \Pi w(k))$
  - $u_{im}(k) = \Gamma w(k)$
    * However this again requires full measurement, which we will address next time
  - The overall design will be $u(k) = K(x(k) - \Pi w(k)) + \Gamma w(k) = Kx(k) + (\Gamma - K\Pi)w(k)$, consisting of a state feedback and a feedforward controller