Lecture 21, Feb 27, 2026

Linear Quadratic Optimal Control

- Consider a linear system $x(k+1) = Ax(k) + Bu(k)$ and a quadratic cost $J(x_0) = \frac{1}{2}\sum_{k=0}^{\infty} x^T(k)Qx(k) + u^T(k)Ru(k)$ where Q is positive semidefinite, R is positive definite
  - Note that here we don't need a forgetting factor; we can show that the positive definiteness of R gives us convergence
- Consider the cost starting at k, $J(x(k)) = \frac{1}{2}\sum_{i=k}^{\infty} x^T(i)Qx(i) + u^T(i)Ru(i)$
- Derive the Bellman equation: $$\begin{aligned} V^\mu(x(k)) &= \frac{1}{2}\sum_{i=k}^{\infty} x^T(i)Qx(i) + \mu^T(i)R\mu(i) \\ &= \frac{1}{2}\left(x^T(k)Qx(k) + \mu^T(k)R\mu(k)\right) + \frac{1}{2}\sum_{i=k+1}^{\infty} x^T(i)Qx(i) + \mu^T(i)R\mu(i) \\ &= \frac{1}{2}\left(x^T(k)Qx(k) + \mu^T(k)R\mu(k)\right) + V^\mu(x(k+1)) \end{aligned}$$
  - Note $\mu(k)$ is a shorthand for $\mu(x(k))$
- We use a linear feedback $\mu(k) = -Kx(k)$, which we can show results in a quadratic optimal cost $V^\mu(x(k)) = \frac{1}{2}x^T(k)Px(k)$ where P is positive definite
  - This implies $$\begin{aligned} x^T(k)Px(k) &= x^T(k)Qx(k) + \mu^T(k)R\mu(k) + x^T(k+1)Px(k+1) \\ &= x^T(k)Qx(k) + \mu^T(k)R\mu(k) + (Ax(k) + B\mu(k))^T P(Ax(k) + B\mu(k)) \\ &= x^T(k)Qx(k) + x^T(k)K^TRKx(k) + x^T(k)(A-BK)^TP(A-BK)x(k) \end{aligned}$$
  - Since this holds for all x, we have $(A-BK)^TP(A-BK) - P + Q + K^TRK = 0$
  - However, this has both K and P as unknowns, so we still need more
- Consider the cost with respect to some $u(k)$, $x^T(k)Px(k) = x^T(k)Qx(k) + u^T(k)Ru(k) + (Ax(k) + Bu(k))^TP(Ax(k) + Bu(k))$; since this is optimal, we can differentiate it with respect to u and set it to zero
  - Doing this we get $Ru(k) + B^TP(Ax(k) + Bu(k)) = 0$
    * Note $\frac{\partial}{\partial u}y^TPy = 2y^TP\frac{\partial y}{\partial u} = 2(Ax+Bu)^TPB$ and $\frac{\partial}{\partial u}u^TRu = 2u^TR$
  - Therefore $Ru(k) + B^TPAx(k) + B^TPBu(k) = 0 \implies -RKx(k) + B^TPAx(k) - B^TPBKx(k) = 0$, and since this must hold for all x, $-RK + B^TPA - B^TPBK = 0 \implies K = (R + B^TPB)^{-1}B^TPA$
    * $R + B^TPB$ is always invertible due to positive definiteness of R and positive semidefiniteness of B^TPB
- Substituting K, we get the *Riccati equation* $A^TPA - P + Q - A^TPB(R + B^TPB)^{-1}B^TPA = 0$
  - Notice that this can be iterated: $P^{j+1} = A^TP^jA + Q - A^TP^jB(R + B^TPB)^{-1}B^TP^jA$, giving us a numerical method to solve for P (value iteration)
- Similar to the continuous time case, we need 3 conditions for a unique positive definite solution to the Riccati equation (Wonham, 1968):
  - (A, B) is stabilizable: this ensures there exists a controller driving all states to zero, and thus a finite cost
  - R is positive definite: this ensures controls stay bounded
  - (Q, A) is detectable: this is required to have a unique solution
- One can further show that the linear controller is the best controller possible, including all nonlinear and time-varying controls