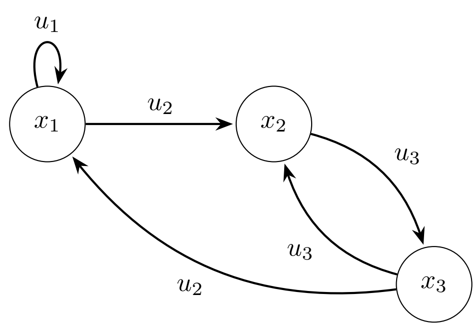
Lecture 18, Feb 13, 2026

Dynamic Programming (Finite Time Horizon)

- Using the principle of optimality, the dynamic programming algorithm computes the optimal cost backwards in time:
  - At each time step, we compute the optimal cost for all possible values of $x(k)$, using the principle of optimality and the optimal cost for all values of $x(k)$ in the next step
  - $J_N(x(N)) = r_N(x(N)), \forall x(N) \in \mathcal{X}$
    - For the final time step, the optimal cost is simply the terminal cost for each possible value of the state
  - $J_k(x(k)) = \min\limits_{u(k)\in\mathcal{U}(x(k))} \{ r(x(k), u(k)) + J_{k+1}(f(x(k), u(k))) \}, \forall x(k) \in \mathcal{X}$
    - For all previous time steps, the optimal cost is obtained by choosing among all possible inputs $u(k)$ for each possible value of the state, using the future costs computed before
  - The optimal input sequence π^* can then be recovered by noting which input we chose at each time step



- Example: consider the finite state discrete time system above, with the terminal cost being zero for all states, and all other costs are 1, except $r(x_1, u_1) = 0$; consider a finite horizon optimal control problem with $N = 2$; compute J_i for $i = 0, 1, 2$ using the dynamic programming algorithm
  - We have $\mathcal{X} = \{ x_1, x_2, x_3 \}$, $u(x_1) = \{ u_1, u_2 \}, u(x_2) = \{ u_3 \}, u(x_3) = \{ u_3, u_2 \}$
  - Starting from $k = 2$, $J_2(x_1) = J_2(x_2) = J_2(x_3) = r_N(x(N)) = 0$ since all terminal costs are zero
  - For $k = 1$:
    - $$\begin{aligned} J_1(x_1) &= \min \{ r(x_1, u_1) + J_2(f(x_1, u_1)), r(x_1, u_2) + J_2(f(x_1, u_2)) \} \\ &= \min \{ 0 + J_2(x_1), 1 + J_2(x_2) \} \\ &= 0 \end{aligned}$$
    - $J_1(x_2) = r(x_2, u_3) + J_2(f(x_2, u_3)) = 1$ since there is only one connection
    - Similarly $J_1(x_3) = 1$
  - For $k = 0$:
    - $$\begin{aligned} J_0(x_1) &= \min \{ r(x_1, u_1) + J_1(x_1), r(x_1, u_2) + J_1(x_2) \} \\ &= \min \{ 0, 1 \} \\ &= 0 \end{aligned}$$
      - This makes sense since the optimal policy starting at x_1 is to stay there forever and get zero cost
    - $$\begin{aligned} J_0(x_2) &= r(x_2, u_3) + J_1(x_3) \\ &= 2 \end{aligned}$$
    - $$\begin{aligned} J_0(x_3) &= \min \{ r(x_3, u_2) + J_1(x_1), r(x_3, u_3) + J_1(x_2) \} \\ &= \min \{ 1, 2 \} \\ &= 1 \end{aligned}$$