Lecture 17, Feb 12, 2026

Optimal Control Problem

- Consider the general discrete-time system $x(k+1) = f(k, x(k), u(k))$ where $x(k) \in \mathcal{X}$, where the state space $\mathcal{X}$ is usually $\mathbb{R}^n$ (but can be finite or infinite)
  - For $k \in 0, 1, \dots, N-1$, we want to select a control $\pi = \{\, u(0), u(1), \dots, u(N-1) \mid u(k) \in \mathcal{U}_k(x(k)) \,\}$, i.e. at each time we can select an input in a set based on our current state
  - Let $\Pi(x(0))$ denote the set of all admissible controls for the initial conditions $x(0)$
- Given $x(0) \in \mathcal{X}$ and $\pi \in \Pi(x(0))$, we define a cost $J^\pi(x(0)) = r_N(x(N)) + \sum_{k=0}^{N-1} r_k(x(k), u(k))$
  - $r_N(x(N)) \geq 0$ is the terminal cost and $r_k(x(k), u(k)) \geq 0$ are stage costs
- Our control objective is to find the optimal cost (*value function*) $J^*(x(0)) = \min_{\pi \in \Pi(x(0))} J^\pi(x(0))$ and an optimal control $\pi^* \in \Pi(x(0))$ such that $J^{\pi^*}(x(0)) = J^*(x(0))$, for each $x(0) \in \mathcal{X}$
  - Note that the optimal cost is unique, but the optimal control is not
  - In classical adaptive control we first find the optimal cost, and then use it to find the optimal control

Theorem

Principle of Optimality: Let $x(0) \in \mathcal{X}$ and suppose $\pi^* = \{\, u^*(0), \dots, u^*(N-1) \,\} \in \Pi(x(0))$, with $\{\, x(0), x^*(1), \dots, x^*(N) \,\}$, then for any $j \in \{\, 1, \dots, N-1 \,\}$, the control $\pi = \{\, u^*(j), u^*(j+1), \dots, u^*(N-1) \,\}$ and sequence of states $\{\, x^*(j), x^*(j+1), \dots, x^*(N) \,\}$ is optimal for the sub-problem with $x^*(j)$ as initial condition.

- Informally, this is stated as "the tail of the optimal control is optimal for the tail sub-problem"