Lecture 16, Feb 10, 2026

Observability and Observers

- Consider the LTI system $\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$ where $x(k) \in \mathbb{R}^n, u(k) \in \mathbb{R}^m, y(k) \in \mathbb{R}^p$
  - To stabilize this system via pole placement, we can use a controller $u(k) = Kx(k)$, but we usually don't have access to the state $x(k)$ and instead must estimate them from the measurements $y(k)$ and inputs $u(k)$
- Given measurements and inputs, can we find the state and initial conditions?
  - $y(k) = C\left(A^k x(0) + \sum_{j=0}^{k-1}(A^{k-1-j}Bu(j))\right) + Du(k)$
  - $\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x(0) + \begin{bmatrix} Du(0) \\ CBu(0) + Du(1) \\ \vdots \\ Du(n-1) + C\sum_{j=0}^{n-2} A^{n-2-j}Bu(j) \end{bmatrix}$
  - Notice that the only unknown is $x(0)$, so we can solve this as a linear system
  - Rearranging, we get $Q_o x(0) = Y$, where $Q_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{(np)\times n}$ is the *observability matrix*
  - Therefore we can solve for a unique initial condition if and only if Q_o has full column rank

Definition

(C, A) is *observable* if $\operatorname{rank}(Q_o) = n$, where the observability matrix is

$$Q_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

- Using Q_o we we can solve for the initial conditions using a batch algorithm, but we need an online estimate of $x(k)$, which requires an *observer*
  - The observer is an LTI system $\begin{cases} \hat{x}(k+1) = A\hat{x}(k) + Bu(k) + L(y(k) - \hat{y}(k)) \\ \hat{y}(k) = C\hat{x}(k) \end{cases}$
    - Note that we dropped the $Du(k)$ term but it could be easily added
  - Let the estimation error $\tilde{x}(k) = \hat{x}(k) - x(k)$
  - $\tilde{x}(k+1) = \hat{x}(k+1) - x(k+1)$
    $= A\hat{x}(k) + Bu(k) + L(Cx(k) - C\hat{x}(k)) - Ax(k) - Bu(k)$
    $= A\tilde{x}(k) - LC\tilde{x}(k)$
    $= (A - LC)\tilde{x}(k)$
  - Therefore if we can select $L \in \mathbb{R}^{n\times p}$ such that $A - LC$ is Schur stable, $\hat{x}(k)$ exponentially converges to the true $x(k)$
- Suppose (A, B) is stabilizable and $u(k) = Kx(k)$ stabilizes the system, then we can implement $u(k) = K\hat{x}(k)$ where $\hat{x}(k)$ is the estimate from the observer
  - Consider the extended state $\begin{bmatrix} x(k) \\ \tilde{x}(k) \end{bmatrix}$

- $x(k+1) = Ax(k) + Bu(k)$
  $= Ax(k) + BK\hat{x}(k)$
  $= Ax(k) + BK(\tilde{x}(k) + x(k))$
  $= (A + BK)x(k) + BK\tilde{x}(k)$
- Therefore $\begin{bmatrix} x(k+1) \\ \tilde{x}(k+1) \end{bmatrix} = \begin{bmatrix} A+BK & BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x(k) \\ \tilde{x}(k) \end{bmatrix}$
- If the eigenvalues of $A + BK$ and $A - LC$ are both stable, then the overall system is stable since the eigenvalues of the larger matrix is the union of the two
- This is known as the *separation principle*

Definition

(C, A) is *detectable* if there exists $L \in \mathbb{R}^{n\times p}$ such that $|\lambda| < 1, \forall\lambda \in \sigma(A - LC)$.

Duality

- Notice that we have $\sigma(A - LC) = \sigma((A - LC)^T) = \sigma(A^T - C^T L^T) = \sigma(A^T + C^T(-L^T))$, which looks exactly like the controllability case
  - We can use the normal pole assignment techniques for $A + BK$, and take $L = -K^T$ to place the poles of the observer
- The controllability matrix of (A^T, C^T) is the same as the observability matrix of (C, A), so we can reuse results from controllability

Definition

Duality Theory:

- (C, A) is observable if and only if (A^T, C^T) is controllable.
- (C, A) is detectable if and only if (A^T, C^T) is stabilizable.
- (C, A) is observable if and only if $\operatorname{rank}\left(\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}\right) = n, \forall\lambda \in \sigma(A)$.
- (C, A) is detectable if and only if $\operatorname{rank}\left(\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}\right) = n, \forall\lambda \in \sigma(A)$ where $|\lambda| \geq 1$.