Lecture 15, Feb 6, 2026

Theoretical Justification of the Dynamic Error Model

- Consider the general dynamic error model: $\begin{cases} x(k+1) = Ax(k) + B\tilde{\psi}^T(k)w(k) \\ e(k) = Cx(k) \end{cases}$, where (A, B) is controllable and (C, A) is observable, and A is Schur stable
  - Let $H(z) = C(zI - A)^{-1}B$
  - Define the error estimate $\hat{e}(k) = H(z)\left[\hat{\psi}^T(k)w(k)\right] - \hat{\psi}^T(k)H(z)I\left[w(k)\right]$
  - Define the augmented regressor $w_a(k) = H(z)I\left[w(k)\right]$
  - Define the augmented error $e_a(k) = e(k) - \hat{e}(k) = \tilde{\psi}^T(k)w_a(k) + \varepsilon(k)$ where $\tilde{\psi}(k) = \hat{\psi}(k) - \psi$ and $\varepsilon(k) \to 0$
- We reuse the adaption law from the static error model: $\hat{\psi}(k+1) = \hat{\psi}(k) - \gamma(k)e_a(k)w_a(k), \gamma(k) = \dfrac{\bar{\gamma}}{1 + \|w_a(k)\|^2}, \bar{\gamma} \in (0, 2)$

Theorem

Swapping Lemma: Let $\hat{\psi} : \mathbb{N}_0 \mapsto \mathbb{R}^q$ and $w : \mathbb{N}_0 \mapsto \mathbb{R}^q$ be discrete signals, and $H(z) = C(zI - A)^{-1}B$ be stable (i.e. A is Schur stable), and (C, A, B) is a minimal realization (i.e. no pole-zero cancellations in the transfer function), then

$$\hat{\psi}^T(k)H(z)I\left[w(k)\right] - H(z)\left[\hat{\psi}^T(k)w(k)\right] = c\tilde{\eta}(k)$$

where $\tilde{\eta}(k)$ has the dynamics

$$\begin{aligned} \eta_1(k+1) &= A\eta_1(k) + Bw^T(k) \\ \tilde{\eta}(k+1) &= A\tilde{\eta}(k) + \eta_1(k+1)\Delta\tilde{\psi}(k) \\ \Delta\tilde{\psi}(k) &= \tilde{\psi}(k+1) - \tilde{\psi}(k) \end{aligned}$$

Theorem

For the system

$$\begin{aligned} x(k+1) &= Ax(k) + B\tilde{\psi}^T(k)w(k) \\ e(k) &= Cx(k) \end{aligned}$$

where A is Schur stable, with the adaptation law

$$\hat{\psi}(k+1) = \hat{\psi}(k) - \gamma(k)e_a(k)w_a(k) \qquad \gamma(k) = \frac{\bar{\gamma}}{1 + \|w_a(k)\|^2}$$

Suppose $\bar{\gamma} \in (0, 2)$ and $w \in l_\infty$, then $\hat{\psi} \in l_\infty, e_a \in l_\infty \cap l_2$, and $e_a(k) \to 0$ and $e(k) \to 0$ as $k \to \infty$.

- Proof:
  - $e_a(k) = \tilde{\psi}^T(k)w_a(k) + \varepsilon(k)$
  - $\tilde{\psi}(k+1) = \tilde{\psi}(k) - \gamma(k)e_a(k)w_a(k)$
  - To handle $\varepsilon(k)$, we need to give it a model; since $\varepsilon(k) \to 0$, we know there exists $(C_\varepsilon, A_\varepsilon)$ such that A_ε is Schur stable and models the decay of $\varepsilon(k)$, i.e. $\|\varepsilon(k)\| \leq \|C_\varepsilon \nu(k)\|$ where $\nu(k+1) = A_\varepsilon \nu(k)$
  - Let P be the positive definite solution to $A_\varepsilon^T P A_\varepsilon - P = -I$
  - Consider the Lyapunov function $V = \tilde{\psi}^T(k)\tilde{\psi}(k) + \beta\nu^T(k)P\nu(k)$, where $\beta > 0$ to be determined, so V is positive definite at $(\tilde{\psi}, \nu) = (0, 0)$

- $\Delta V = \cdots$

  $= -2\gamma(k)e_a(k)w_a^T(k)\tilde{\psi}(k) - \beta\|\nu(k)\|^2 + \gamma^2(k)e_a^2(k)\|w_a(k)\|^2$

  $= \cdots$

  $= -2\gamma(k)e_a^2(k) + 2\gamma(k)e_a(k)\varepsilon(k) - \beta\|\nu\|^2 + \gamma^2(k)e_a^2(k)\|w_a(k)\|^2$
  - The first and last terms are the same as the static error model
  - The second terms is a cross term representing the coupling resulting from $\varepsilon(k)$
- Use Young's inequality: $2\gamma(k)e_a(k)\varepsilon(k) \leq \gamma^2(k)e_a(k) + \varepsilon^2(k)$
  - After breaking down the cross term and combining it into the other terms, we can conclude that $\Delta V \leq 0$ for sufficiently large $\beta \geq 0$
- Reusing the static error model arguments, we have $\hat{\psi}, \tilde{\psi}, e_a \in l_\infty$, then $e_a \in l_2$ which implies $e_a(k) \to 0$, and $\Delta\tilde{\psi}(k) = \tilde{\psi}(k+1) - \tilde{\psi}(k) \to 0$