Lecture 14, Feb 5, 2026

Theoretical Justification of the Static Error Model

- Consider the static error model: $\begin{cases} \hat{\psi}(k+1) = \hat{\psi}(k) - \gamma(k)e(k)w(k) \\ e(k) = (\hat{\psi}(k) - \psi)^T w(k) = \tilde{\psi}^T(k)w(k) \end{cases}$
  - $$\begin{aligned} \tilde{\psi}(k+1) &= \hat{\psi}(k+1) - \psi \\ &= \hat{\psi}(k) - \gamma(k)e(k)w(k) - \psi \\ &= \tilde{\psi}(k) - \gamma(k)e(k)w(k) \\ &= \tilde{\psi}(k) - \gamma(k)(\tilde{\psi}^T(k)w(k))w(k) \\ &= \tilde{\psi}(k) - \gamma(k)w(k)w^T(k)\tilde{\psi}(k) \\ &= (I - \gamma(k)w(k)w^T(k))\tilde{\psi}(k) \\ &= A_{cl}(k)\tilde{\psi}(k) \end{aligned}$$
  - We now have a linear time-varying system
- We can show that the above is stable for $\bar{\gamma} \in (0, 2)$, and $\hat{\psi} \in l_\infty, e \in l_\infty$
  - Consider the Lyapunov function $V = \|\tilde{\psi}(k)\|^2 = \tilde{\psi}^T(k)\tilde{\psi}(k)$
  - $$\begin{aligned} \Delta V &= V(k+1) - V(k) \\ &= \tilde{\psi}^T(k)(I - \gamma(k)w(k)w^T(k))^T(I - \gamma(k)w(k)w^T(k))\tilde{\psi}(k) - \tilde{\psi}^T(k)\tilde{\psi}(k) \\ &= \tilde{\psi}^T(k)\left(-2\gamma(k)w(k)w^T(k) + \gamma^2(k)\|w(k)\|^2 w(k)w^T(k)\right)\tilde{\psi}(k) \\ &= -\gamma(k)\left(2 - \gamma(k)\|w(k)\|^2\right)\tilde{\psi}^T(k)w(k)w^T(k)\tilde{\psi}(k) \\ &= -\gamma(k)\left(2 - \gamma(k)\|w(k)\|^2\right)e^2(k) \\ &= -\gamma(k)\left(2 - \frac{\bar{\gamma}\|w(k)\|^2}{1 + \|w(k)\|^2}\right)e^2(k) \end{aligned}$$
  - Since $\dfrac{\|w(k)\|^2}{1 + \|w(k)\|^2} < 1$, $\Delta V \leq -\gamma(k)(2 - \bar{\gamma})e^2(k)$
    - Therefore we need $\gamma \in (0, 2)$ for ΔV to be strictly negative
  - Since $w(k) \in l_\infty$, there exists $c_0 > 0$ such that $\gamma(k)(2 - \bar{\gamma}) \geq c_0, \forall k \geq 0$, and therefore $\Delta V \leq -c_0\|e(k)\|^2 < 0$
  - By Lyapunov, $\tilde{\psi} = 0$ is stable, so $\tilde{\psi} \in l_\infty$, which means $\hat{\psi} \in l_\infty$ and $e \in l_\infty$
- Furthermore, we can show that $e \in l_2$, and as a consequence $e(k) \to 0$ as $k \to 0$
  - Since $\Delta V < 0$ and $V \geq 0$, by the monotone convergence theorem $V(\tilde{\psi}(k)) = V(k)$ has a limit; let $\lim_{k\to\infty} V(k) = V_\infty$
  - $$\begin{aligned} &\Delta V \leq -c_0 e^2(k) < 0 \\ \implies &c_0 e^2(k) \leq v(k) - v(k+1) \\ \implies &c_0 \sum_{k=0}^{\infty} e^2(k) \leq \sum_{k=0}^{\infty}(V(k) - V(k+1)) \\ \implies &c_0 \sum_{k=0}^{\infty} e^2(k) \leq V(0) - V_\infty \end{aligned}$$
  - Since V_∞ exists, $e(k) \in l_2$, and consequently $e(k) \to 0$ as $k \to 0$ (since otherwise the sum would diverge)