Lecture 1, Jan 6, 2026

Discrete-Time Systems

- Some physical phenomena is best captured with discrete-time dynamics rather than continuous time, e.g. the human saccadic system (where the eye moves around rapidly to sample different objects)
- (LTI) Discrete processes can be modelled in 3 ways:
  - State models: $x(k+1) = Ax(k) + Bu(k)$ where:
    $$y(k) = Cx(k) + Du(k)$$
    * k is a discrete sample time, which is not necessarily evenly spaced
    * $x(k)$ is the state
    * $u(k)$ is the input
    * $y(k)$ is the output
  - Difference equations: $y(k) + a_1 y(k-1) + \cdots + a_n y(k-n) = b_0 u(k) + b_1 u(k-1) + \cdots + b_m u(k-m)$
    * Written in terms of the input, output, and delayed values (as opposed to always a single shift in k as in the state model)
    * The difference equation is to the state-space model the same way that an ODE is to a continuous time state-space model
  - Transfer functions: $\dfrac{Y(z)}{U(z)}$ where $Y(z)$, $U(z)$ are the Z-transformed versions of the output and input
    * Analogous to continuous time transfer functions, except using Z-transforms instead of Laplace transforms
- Example: modelling the saccade process: let the state $x(k) \in \mathbb{R}$ be the change in eye angle at the end of the saccade, the error $e(k) = r(k) - x(k)$ representing the difference between the desired change in eye angle and the actual eye angle
  - How does the brain ensure that the saccade is always correct, despite various disturbances to the system (e.g. glasses)?
  - Augment the error with a disturbance $d(k)$, such that $e(k) = r(k) + d(k) - x(k)$
  - Consider a simple plant model $x(k+1) = u(k)$, which is in the form of a state model
    * The TF is $zX(z) = U(z) \implies \dfrac{X(z)}{U(z)} = \dfrac{1}{z}$
      - For Z-transforms, a shift $x(k) \to x(k+1)$ becomes a multiply, $X(z) \to zX(z)$, analogous to how differentiation becomes multiplication for Laplace transforms
    * The difference equation is obtained by simply shifting back, $x(k) = u(k-1)$
      - In this case the difference between state model and difference equation is trivial, but in higher dimension systems the difference equation becomes higher order
  - Derive the error model, $e(k+1) = r(k+1) + d(k+1) - x(k+1)$
    $$= \bar{r} + \bar{d} - x(k+1)$$
    $$= \bar{r} + \bar{d} - u(k)$$
    * We assume $r(k) = r(k+1) = \bar{r}$ is a constant target, and $d(k) = d(k+1) = \bar{d}$ is a constant disturbance
    * To get the error model, we keep forward shifting the dynamics until the input $u(k)$ appears
    * We only needed to forward shift once, so this is a first-order model
  - Consider a controller $u(k) = u_s(k) + u_m(k)$ where u_m is the internal model, which handles the steady state behaviour, and u_s gives us good transient behaviour
  - Choose $u_s(k) = Ke(k)$ for some gain K, so $e(k+1) = -Ke(k) - u_m(k) + \bar{d} + \bar{r}$
    * Suppose $u_m(k) = 0$; is this stable?
      - For a system in canonical form $x(k+1) = Ax(k)$, the system is asymptotically stable if and only if the eigenvalues of A are inside the unit circle, so in this case the undriven system is stable if $|K| < 1$
      - To find the steady state (we know it exists in this case), set $e_{ss} = e(k) = e(k+1) \implies e_{ss} = \dfrac{1}{1+K}(\bar{d} + \bar{r})$
      - Notice that the higher the gain, the smaller our steady state error, but we can never

eliminate the steady state error if $u_m(k) = 0$ since we can only set K to be so high before the system becomes unstable

– We add an internal model $\hat{w}(k+1) = \hat{w}(k) + Ge(k)$ and $u_i(k) = \phi\hat{w}(k)$, and we can show that the resulting closed-loop system is asymptotically stable

Note

The failure of the controller with $u_m(k) = 0$ is an example of the *Internal Model Principle*, which intuitively states that "any good regulator must create an internal model of the dynamic structure of the environment in the closed loop system," i.e. the controller must incorporate a model of the *exosystem*, which is capable of producing any disturbance frequencies that could enter the system.