1 Dynamical Systems

Difference Equations: For degree n system, $n \le m$ for causality:
$$y(k) + a_1 y(k-1) + \cdots + a_n y(k-n) = b_0 u(k) + \cdots + b_m u(k-m)$$

Z-Transform: $\mathcal{Z}\{x(k)\} = X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}, z \in \mathbb{C}$

Fwd. Shift: $\mathcal{Z}\{x(k+m)\} = z^m X(z)$ (no IC); $\mathcal{Z}\{x(k+1)\} = zX(z) - zX(0)$

SS to TF: $G(z) = C(zI - A)^{-1}B + D \in \mathbb{R}^{p \times m}$.

TF to SS: $G(z) = N(z)/D(z) \implies D(z)V(z) = U(z), N(z)V(z) = Y(z)$; inverse Z-transform to get DE, and let $x(k) = \begin{bmatrix} v(k) & v(k+1) & \cdots \end{bmatrix}^T$, rearrange to get state eqn. from first and measurement eqn. from second.

Time Response: $x(k) = A^k x(0) + \sum_{i=0}^{k-1} A^{k-1-i} B u(i)$, consisting of initial state response and input response (*not* transient and steady-state).

Matrix Power: $A^k = \mathcal{Z}^{-1}\left\{(zI - A)^{-1} z\right\} x(0) = P \Lambda^k P^{-1}, P^{-1}AP = \Lambda$.

Poles & Eigenvalues: All poles are eigenvalues, but not all eigenvalues are poles (pole-zero cancellation); a stable transfer function does not imply stable states.

Qualitative Behaviour: Real distinct poles: $\mathcal{Z}^{-1}\left\{\frac{z}{z-p}\right\} = p^k$; decay if $|p| < 1$, steady if $|p| = 1$, unstable if $|p| > 1$, alternating if $p < 0$.

Conjugate poles: $\mathcal{Z}^{-1}\left\{\frac{z^2}{(z - re^{j\omega})(z - re^{-j\omega})}\right\} = \frac{1}{\sin \omega} r^k \sin(k\omega + \phi)$; r is rate of decay (smaller is faster, $r > 1$ unstable), ω is oscillation frequency.

2 Stability

Equilibrium: $\bar{x}$ is equilibrium of $x(k+1) = f(k, x(k))$ if $\bar{x} = f(k, \bar{x})$.

Stability: $\forall k_0 \ge 0, \varepsilon > 0, \exists \delta(\varepsilon, k_0) > 0$ s.t. $\|x_0 - \bar{x}\| < \delta \implies \|x(k; k_0, x_0) - \bar{x}\| < \varepsilon, \forall k \ge k_0$.

Asymptotic Stability: Stable and $\exists \delta(k_0) > 0$ s.t. $\|x_0 - \bar{x}\| < \delta \implies \lim_{k \to \infty} x(k; k_0, x_0) = \bar{x}$ (attractive).

Uniform Asymptotic Stability: A.S. and δ does not depend on k_0.

Global Asymptotic Stability: A.S. and $\delta(k_0)$ can be arbitrary large, i.e. all initial conditions converge to $\bar{x}$. G.U.A.S. if also U.A.S.

Exponential Stability: $\exists c, \delta > 0, \lambda \in (0,1)$ s.t. $\|x(k_0) - \bar{x}\| < \delta \implies \|x(k) - \bar{x}\| \le c\|x(k_0)\| \lambda^{k - k_0}, \forall k \ge k_0$. Implies U.A.S.; G.E.S. if δ can be arbitrarily large.

Positive (Semi)Definite: $V : \mathbb{R}^n \mapsto \mathbb{R}$ is positive definite at $x = 0$ if $V(0) = 0$ and $V(x) > 0, \forall x \ne 0$; or positive semidefinite if $V(0) = 0$ and $V(x) \ge 0, \forall x$.

Class $\mathcal{K}_\infty$: $\kappa : [0, \infty) \mapsto [0, \infty)$ is class $\mathcal{K}_\infty$ if $\kappa(0) = 0$, $\lim_{x \to \infty} \kappa(s) \to \infty$, and strictly increasing on $[0, \infty)$.

Radial Unboundedness: $\exists \kappa \in \mathcal{K}_\infty$ s.t. $V(k, x) \ge \kappa(\|x\|), \forall k \ge 0, x \in \mathbb{R}^n$. Usually can just show $\exists c_1 > 0$ s.t. $V(k, x) \ge c_1 \|x\|^2$ (stricter condition).

Forward Difference: For time-invariant systems, $\Delta V(x) = V(f(x)) - V(x)$; For time-varying systems, $\Delta V(k, x) = V(k+1, f(k, x(k))) - V(k, x(k))$.

Main Theorem of Lyapunov: Let the *Lyapunov function* $V : \mathbb{N}_0 \times \mathbb{R}^n \mapsto \mathbb{R}$ positive definite such that $V(k, 0) = 0, \forall k$. If $\Delta V(k, x)$ is negative semidefinite in x, then $\bar{x} = 0$ is stable. If $\Delta V(k, x)$ is negative definite in x, then $\bar{x} = 0$ is A.S.; if V is also radially unbounded, then $\bar{x} = 0$ is G.A.S.

Exponential Stability Theorem: Let $\mathcal{D} \subseteq \mathbb{R}^n$ with $\bar{x} = 0 \in \mathcal{D}$; $\bar{x}$ is E.S. $\iff \exists V : \mathbb{N}_0 \times \mathbb{R}^n \mapsto \mathbb{R}$ and $c_1, c_2, c_3 > 0$ s.t. $c_1\|x\|^2 \le V(k, x) \le c_2 \|x\|^2$ and $\Delta V(k, x) \le -c_3 \|x\|^2, \forall k \le 0, x \in \mathcal{D}$. If $\mathcal{D} = \mathbb{R}^n$, then $\bar{x} = 0$ is G.E.S.

Schur Stable: A is Schur stable if $|\lambda| < 1, \forall \lambda \in \sigma(A)$, i.e. all eigenvalues of A are inside the open unit disk in $\mathbb{C}$.

Stability of LTI Systems: For $x(k+1) = Ax(k)$, $\bar{x} = 0$ is G.E.S. if and only if A is Schur stable; $\bar{x} = 0$ is stable if and only if $|\lambda| \le 1, \forall \lambda \in \sigma(A)$, and any eigenvalues $|\lambda| = 1$ have a Jordan block of size 1.

Lyapunov Equation (LTI): $\bar{x} = 0$ is G.E.S. iff $\forall Q \in \mathbb{R}^{n \times n}, Q = Q^T, Q \succ 0$, there exists a unique $P \in \mathbb{R}^{n \times n}, P = P^T, P \succ 0$ s.t. $A^T P A - P = -Q$. G.E.S. can be shown with only one Q using $V(x) = x^T P x, \Delta V(x) = -x^T Q x$, and that $\lambda_{\min}(P)\|x\|^2 \le x^T P x \le \lambda_{\max}(P)\|x\|^2$.

3 State & Output Feedback Stabilization

Controllability: The LTI system (A, B) is controllable if $\operatorname{rank}(Q_c) = n$, where the controllability matrix $Q_c = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$.

Pole Placement Theorem: If (A, B) is controllable, then for any desired symmetric spectrum $\{\lambda_{1d}, \ldots, \lambda_{nd}\}, \lambda_{id} \in \mathbb{C}$, there exists a state feedback $u(k) = Kx(k)$ with $K \in \mathbb{R}^{m \times n}$, such that $\sigma(A + BK) = \{\lambda_{1d}, \ldots, \lambda_{nd}\}$.

Controllable Canonical Form: A single-input system (A, B) is controllable if and only if there is a coordinate transform $P = Q_c T, z = P^{-1}x$ where the resulting system $z(k+1) = P^{-1}APz(k) + P^{-1}Bu(k) = \tilde{A}z(k) + \tilde{B}u(k)$ has the form
$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad T = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$
where a_i are coefficients in $\det(sI - A)$.

Deadbeat Control: If (A, B) is controllable, then by assigning $\sigma(A + BK)$ to all zeros, $x(k) \to 0$ in exactly n steps, i.e. $x(n) = x(n+1) = \cdots = 0$.

Stabilizability: (A, B) is stabilizable if $\exists K \in \mathbb{R}^{m \times p}$ s.t. $\sigma(A + BK)$ is in the open unit disk in $\mathbb{C}$, i.e. the closed-loop system is asymptotically stable.

Controllable Eigenvalue: $\lambda \in \sigma(A)$ is controllable for the system (A, B) if $\operatorname{rank}\left(\begin{bmatrix} A - \lambda I & B \end{bmatrix}\right) = n$.

PBH Test: (A, B) is controllable if and only if every $\lambda \in \sigma(A)$ is controllable. (A, B) is stabilizable if and only if every $|\lambda| \ge 1$ is controllable.

Observability: (C, A) is observable if $\operatorname{rank}(Q_o) = n$, where the observability matrix is $Q_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$. This is equivalent to Q_c^T for the dual system (A^T, C^T).

Detectability: (C, A) is detectable if there exists $L \in \mathbb{R}^{n \times p}$ such that $|\lambda| < 1, \forall \lambda \in \sigma(A - LC)$.

Duality: The system (C, A) is:

1. Observable if and only if (A^T, C^T) is controllable.
2. Detectable if and only if (A^T, C^T) is stabilizable.
3. Observable if and only if $\operatorname{rank}\left(\left[\begin{smallmatrix} A - \lambda I \\ C \end{smallmatrix}\right]\right) = n, \forall \lambda \in \sigma(A)$.
4. Detectable if and only if $\operatorname{rank}\left(\left[\begin{smallmatrix} A - \lambda I \\ C \end{smallmatrix}\right]\right) = n, \forall \lambda \in \sigma(A)$ where $|\lambda| \ge 1$.

State Observer: $\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + L(y(k) - \hat{y}(k)), \hat{y}(k) = C\hat{x}(k)$; the estimation error $\tilde{x}(k) = \hat{x}(k) - x(k)$ has dynamics $\tilde{x}(k+1) = (A - LC)\tilde{x}(k)$.

Separation Principle: To stabilize (A, B, C) through output feedback, we can separately design a state feedback stabilizer and observer, then $u(k) = K\hat{x}(k)$ asymptotically stabilizes the system. The closed-loop system has dynamics $\begin{bmatrix} x(k+1) \\ \tilde{x}(k+1) \end{bmatrix} = \begin{bmatrix} A + BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x(k) \\ \tilde{x}(k) \end{bmatrix}$.

4 Adaptive Control

l_∞-Norm: $\|x\|_{l_\infty} = \sup_{k \ge 0} \|x(k)\|$; denote $x(k) \in l_\infty$ if $\|x\|_{l_\infty}$ exists.

l_2-Norm: $\|x\|_{l_2} = \sqrt{\sum_{k=0}^{\infty} \|x(k)\|^2}$; denote $x(k) \in l_2$ if $\|x\|_{l_2}$ exists.

Persistent Excitation (PE): $w(k) \in \mathbb{R}^q$ is PE if $\exists \beta_0 > 0, N \in \mathbb{N}$ and $N > 0$ s.t. $\beta_0 I \preccurlyeq W(k, N) = \frac{1}{N} \sum_{\tau = k}^{k+N-1} w(\tau) w^T(\tau), \forall k \in \mathbb{N}_0$ where $A \preccurlyeq B$ denotes $x^T A x \le x^T B x, \forall x \in \mathbb{R}^q$.

Gradient Law: $\hat{\psi}(k+1) = \hat{\psi}(k) - \gamma(k) e(k) w(k), \gamma(k) = \frac{\bar{\gamma}}{1 + \|w(k)\|^2}, \bar{\gamma} \in (0, 2)$. $e(k) \to 0$ is guaranteed but $\hat{\psi}(k) \to \psi$ requires the regressor be PE.

Static Error Model: Given measurements $y(k) \in \mathbb{R}$, known regressor $w(k) \in \mathbb{R}^q$, linear model $y(k) = \psi^T w(k)$, recover unknown $\psi \in \mathbb{R}^q$.

Define parameter estimate $\hat{\psi}(k)$, output estimate $\hat{y} = \hat{\psi}^T(k) w(k)$, and prediction error $e(k) = \hat{y}(k) - y(k) = \hat{\psi}^T(k) w(k) - \psi^T w(k) = \tilde{\psi}^T(k) w(k)$. Update $\hat{\psi}(k)$ using gradient law.

Error dynamics $\tilde{\psi}(k+1) = (I - \gamma(k) w(k) w^T(k)) \tilde{\psi}(k)$. Assuming $w \in l_\infty$, then $\tilde{\psi}(k) = 0$ A.S. by Lyapunov with $V = \|\tilde{\psi}(k)\|^2, \Delta V = -\gamma(k)\left(2 - \frac{\bar{\gamma}\|w(k)\|^2}{1 + \|w(k)\|^2}\right) e^2(k)$. $\tilde{\psi} \in l_\infty$, $\hat{\psi} \in l_\infty$, $e \in l_2$, $e(k) \to 0$.

Model Reference Adaptive Control (MRAC): Given system $x(k+1) = Ax(k) + Bu(k)$, reference $x_r(k+1) = A_r x_r(k) + B_r r(k)$, with known A, B, A_r, Schur stable A_r, unknown B_r, assume $r(k) = \psi^T w(k)$, goal is $x(k) \to x_r(k)$.

Use controller $u(k) = Kx(k) + \hat{\psi}^T(k) w(k)$ for $K \in \mathbb{R}^{1 \times n}$, estimate $\hat{\psi}(k) \in \mathbb{R}^q$.

By imposing matching conditions $A_r = A + BK, B_r = bB, b \in \mathbb{R}$, the tracking error $\tilde{x}(k) = x(k) - x_r(k)$ dynamics are $\tilde{x}(k+1) = A_r \tilde{x}(k) + B(\hat{\psi}^T(k) - b\psi^T) w(k)$ which converges if $\hat{\psi}(k) \to b\psi$.

Dynamic Error Model: Given measurable error state $x_e(k)$ with model $x_e(k+1) = Ax_e(k) + B\tilde{\psi}^T(k) w(k)$ with known A, B, Schur stable A, known regressor $w(k)$, let $e(k) = B^T P x_e(k)$ where $P = P^T, P \succ 0, A^T P A - P = I$; goal is $\tilde{\psi}(k) = \hat{\psi}(k) - \psi(k) \to 0$.

Let $H(z)$ be the system transfer function so $e(k) = H(z)\left[\tilde{\psi}^T(k) w(k)\right]$. Let $\hat{y}(k) = H(z)\left[\hat{\psi}^T(k) w(k)\right]$, then $e(k) = \hat{y}(k) - \psi^T H(z) I\,[w(k)]$, and let $\hat{e}(k) = \hat{y}(k) - \hat{\psi}^T(k) H(z) I\,[w(k)]$. Let the augmented regressor $w_a(k) = H(z) I\,[w(k)]$, and the augmented error $e_a(k) = e(k) - \hat{e}(k) = \tilde{\psi}^T(k) w_a(k) + \varepsilon(k)$ where $\varepsilon(k) \to 0$. Now use gradient law with $e_a(k), w_a(k)$ in place of the normal error and regressor.

Assuming $w \in l_\infty$, then $\hat{\psi} \in l_\infty, e_a \in l_\infty \cap l_2$, and $e_a(k), e(k) \to 0$.

Swapping Lemma: Let discrete signals $\hat{\psi} : \mathbb{N}_0 \mapsto \mathbb{R}^q$ and $w : \mathbb{N}_0 \mapsto \mathbb{R}^q$, and transfer function $H(z) = C(zI - A)^{-1}B$ be stable with no pole-zero cancellations, then $\hat{\psi}^T(k) H(z) I\,[w(k)] - H(z)\left[\hat{\psi}^T(k) w(k)\right] = c\tilde{\eta}(k)$ where $\eta_1(k+1) = A\eta_1(k) + Bw^T(k), \tilde{\eta}(k+1) = A\tilde{\eta}(k) + \eta_1(k+1)(\tilde{\psi}(k+1) - \tilde{\psi}(k))$.

5 Optimal Control

Finite-Time Optimal Control: Given a system $x(k+1) = f(k, x(k), u(k))$ where $x(k) \in \mathcal{X}$, for $k \in 0, 1, \dots, N-1$, we want to select a control $\pi = \{\, u(0), u(1), \dots, u(N-1) \mid u(k) \in \mathcal{U}_k(x(k)) \,\} \in \Pi(x(0))$ to minimize the cost

$$J^\pi(x(0)) = r_N(x(N)) + \sum_{k=0}^{N-1} r_k(x(k), u(k)).$$

We wish to find the value function $J^*(x(0)) = \min_{\pi \in \Pi(x(0))} J^\pi(x(0))$ and an optimal control $\pi^* \in \Pi(x(0))$ such that $J^{\pi^*}(x(0)) = J^*(x(0))$, for each $x(0) \in \mathcal{X}$.

Principle of Optimality: Suppose $\pi^* = \{\, u^*(0), \dots, u^*(N-1) \,\} \in \Pi(x(0))$, with states $\{\, x(0), x^*(1), \dots, x^*(N) \,\}$, then for any $j \in \{1, \dots, N-1\}$, the control $\pi = \{\, u^*(j), u^*(j+1), \dots, u^*(N-1) \,\}$ and sequence of states $\{\, x^*(j), x^*(j+1), \dots, x^*(N) \,\}$ is optimal for the sub-problem with $x^*(j)$ as initial condition.

Dynamic Programming (Finite Time Horizon): At each time step, compute the optimal cost $\forall x(k) \in \mathcal{X}$, using the principle of optimality and the optimal cost in the next step: $J_N(x(N)) = r_N(x(N)), J_k(x(k)) = \min_{u \in \mathcal{U}(x(k))} \{\, r(x(k), u) + J_{k+1}(f(x(k), u)) \,\}$. The optimal input sequence π^* is recovered by noting which input we chose at each time step.

Infinite-Time Optimal Control: Select a policy $\pi = \{\, \mu_0, \mu_1, \dots \,\}$ to minimize the cost $V^\pi(x_0) = \sum_{k=0}^{\infty} \gamma^k r(x(k), \mu_k(x(k)))$ where $\gamma \in (0,1)$ is the forgetting factor. The optimal cost is $V^*(x_0) = \inf_{\pi \in \mathcal{P}} \{\, V^\pi(x_0) \,\}$.

HJB Equation: $V^*(x_0) = \inf_{u \in \mathcal{U}(x_0)} \{\, r(x_0, u) + \gamma V^*(f(x_0, u)) \,\}$.

Denote $(T^\mu V)(x) = r(x, \mu(x)) + \gamma V(f(x, \mu(x)))$ so $V^* = TV^*$.

Bellman Equation: $V^\mu(x_0) = r(x_0, \mu(x_0)) + \gamma V^\mu(f(x_0, \mu(x_0)))$ (stationary policy μ). Denote $(T^\mu V)(x) = r(x, \mu(x)) + \gamma V(f(x, \mu(x)))$ so $V^\mu = T^\mu V^\mu$.

Value Iteration: Initialize with any $V^0 \geq 0$. Until convergence, do $V^{j+1} = TV^j \iff V^{j+1}(x) = \inf_{u \in \mathcal{U}(x)} \{\, r(x,u) + \gamma V^j(f(x,u)) \,\}$.

Policy Iteration: Initialize with any admissible feedback $\mu^0 = \mathcal{M}$. Repeat: 1. Policy evaluation: solve $V^{\mu^j}(x) = r(x, \mu^j(x)) + \gamma V^{\mu^j}(f(x, \mu^j(x))), \forall x \in \mathcal{X}$. 2. Policy improvement: $\mu^{j+1}(x) = \arg\min_{u \in \mathcal{U}(x)} \{\, r(x,u) + \gamma V^{\mu^j}(f(x,u)) \,\}$.

LQR Control: For an LTI system $x(k+1) = Ax(k) + Bu(k)$, quadratic cost $J(x_0) = \frac{1}{2} \sum_{k=0}^{\infty} x^T(k) Q x(k) + u^T(k) R u(k)$ where $Q = Q^T, Q \succcurlyeq 0$, $R = R^T, R \succ 0$. $V^\mu(x(k)) = \dfrac{1}{2}\left(x^T(k) Q x(k) + \mu^T(k) R \mu(k)\right) + V^\mu(x(k+1))$.

For $\mu(x(k)) = -Kx(k)$, $V^\mu(x(k)) = \frac{1}{2} x^T(k) P x(k)$ where $P = P^T, P \succ 0$. The Bellman equation reduces to $(A - BK)^T P (A - BK) - P + Q + K^T R K = 0$.

Given value function, $K = (R + B^T P B)^{-1} B^T P A$. Substitute to get discrete algebraic Riccati equation: $P = A^T P A + Q - A^T P B (R + B^T P B)^{-1} B^T P A$.

Unique positive definite solution for P exists if (A, B) stabilizable and Q, A detectable.

6 Reinforcement Learning

Goal: Learn the optimal stationary policy $\mu^*(x)$ online and without knowing the system $x(k+1) = f(x(k), u(k))$.

Value Function Approximation: $V^\mu(x) = \psi^T w(k)$

Q-Function: $Q^\mu(x,u) = r(x,u) + \gamma V^\mu(f(x,u))$, satisfies $Q^\mu(x,u) = r(x,u) + \gamma Q^\mu(f(x,u), \mu(f(x,u)))$. $V^*(x) = \min_{u \in \mathcal{U}} Q^*(x,u)$, $\mu^*(x) = \arg\min_{u \in \mathcal{U}} Q^*(x,u)$.

Q-Function VI: Initialize with any $Q^0 \geq 0$. Until convergence, do $Q^{j+1}(x,u) = r(x,u) + \min_{u' \in \mathcal{U}} \gamma Q^j(f(x,u), u')$.

Q-Function PI: Initialize with any admissible feedback $\mu^0 = \mathcal{M}$. Repeat: 1. Policy evaluation: $Q^{\mu^j}(x,u) = r(x,u) + \gamma Q^{\mu^j}(f(x,u), \mu^j(f(x,u))), \forall x \in \mathcal{X}, u \in \mathcal{U}(x)$. 2. Policy improvement: $\mu^{j+1}(x) = \arg\min_{u \in \mathcal{U}(x)} Q^{\mu^j}(x,u)$.

TD Error: $e(k) = r(x(k), \mu(x(k))) + \gamma V^\mu(x(k+1)) - V^\mu(x(k))$

Q-Learning: Approximate $Q^\mu(x,u) = \psi^T w(x,u)$ and use the fact that $Q^\mu(x(k), u(k)) - \gamma Q^\mu(x(k+1), \mu(x(k+1))) - r(x(k), u(k)) = e(k) = 0$. Let $v(k) = w(x(k), u(k)) - \gamma w(x(k+1), \mu(x(k+1)))$ then $\psi^T v(k) = r(x(k), u(k))$.

Policy evaluation: Let prediction error $e_1(k) = (\hat{\psi}^{j+1})^T(k) v(k) - r(x(k), u(k))$ and new regressor $v(k) = w(x(k), u(k)) - \gamma w(x(k+1), \mu^j(x(k+1)))$. Update using gradient law $\hat{\psi}^{j+1}(k+1) = \hat{\psi}^{j+1}(k) - \gamma_1(k) e_1(k) v(k)$ until convergence of $\hat{\psi}^{j+1}$. Add probing noise to $u(k)$ for PE, so $\mu(x(k)) \neq u(k)$.

Policy improvement: Solve $\frac{\partial Q^\mu(x,u)}{\partial u} = \frac{\partial}{\partial u}\left(\hat{\psi}^{j+1} w(x,u)\right) = 0$ for u as a function of x to get μ^{j+1}.

7 Regulator Design

Regulator Problem: For $x(k) \in \mathbb{R}^n, u(k) \in \mathbb{R}^m, e(k) \in \mathbb{R}^p, w(k) \in \mathbb{R}^q$ and $x(k+1) = Ax(k) + Bu(k) + Ew(k)$, $w(k+1) = Sw(k)$, $e(k) = Cx(k) + Dw(k)$ design regulator $u(k)$ such that: 1. The unforced closed-loop system ($w(k) \equiv 0$) has an A.S. equilibrium, and 2. When $w(0) \neq 0$, $\lim_{k \to \infty} e(k) = 0$.

Regulator Equations: $\Pi S = A\Pi + B\Gamma + E$, $0 = C\Pi + D$, where $x_{ss}(t) = \Pi w(k)$, $u_{ss}(k) = \Gamma w(k)$ gives steady-state solutions for $e(k) = 0$.

Solution exists if (A, B) controllable, (C, A) observable, and $\forall \lambda \in \sigma(S)$, $|\lambda| = 1$, λ has algebraic multiplicity 1, and $\det\left[\begin{smallmatrix} A - \lambda I & B \\ C & 0 \end{smallmatrix}\right] \neq 0$ (nonresonance).

Coord. change: $z(k) = x(k) - \Pi w(k) \Rightarrow \begin{cases} z(k+1) = Az(k) + Bu(k) - B\Gamma w(k) \\ e(k) = Cz(k) \end{cases}$

Controller: $u(k) = u_s(k) + u_{im}(k) = Kz(k) + \Gamma w(k) = Kx(k) + (\Gamma - K\Pi) w(k)$

Partial Measurement: Assume $\left(\left[\begin{smallmatrix} C & D \end{smallmatrix}\right], \left[\begin{smallmatrix} A & E \\ 0 & S \end{smallmatrix}\right]\right) = (C_c, A_c)$ observable; Build observer: $\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + E\hat{w}(k) + L_1(e(k) - \hat{e}(k))$, $\hat{w}(k+1) = S\hat{w}(k) + L_2(e(k) - \hat{e}(k))$, $\hat{e}(k) = C\hat{x}(k) + D\hat{w}(k)$.

Estimation error dynamics $\left[\begin{smallmatrix} \tilde{x}(k+1) \\ \tilde{w}(k+1) \end{smallmatrix}\right] = \left(A_c - \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right] C_c\right) \left[\begin{smallmatrix} \tilde{x}(k) \\ \tilde{w}(k) \end{smallmatrix}\right]$.

8 Adaptive Regulation

Adaptive Regulator Problem: $x(k+1) = Ax(k) + Bu(k) + E\xi(k)$, $\xi(k+1) = S\xi(k)$, $e(k) = Cx(k) + D\xi(k)$ where S, E, D are unknown.

Nikiforov Canonical Representation: Let $d(k) = \Gamma\xi(k)$. Assuming regulator equations are solvable, then for all controllable (F, G) where F Schur stable, if (Γ, S) observable, $\sigma(F) \cap \sigma(S) = \varnothing$, then $\exists M$ s.t. $MS = FM + G\Gamma$ and M invertible. Let $w(k) = M\xi(k)$ so $d(k) = \Gamma M^{-1} w(k) = \psi^T w(k)$, then:

$$w(k+1) = Fw(k) + Gd(k) = (F + G\psi^T) w(k) \qquad d(k) = \psi^T w(k)$$

Disturbance Observer: $\hat{z}_d(k+1) = A\hat{z}_d(k) + Bu(k) + L_d(e(k) - C\hat{z}_d(k))$, $A - L_d C$ Schur stable. Error $\tilde{z}_d(k) = \hat{z}_d(k) - z(k)$, $\tilde{z}_d(k+1) = A_d \tilde{z}_d(k) + Bd(k)$ where $A_d = A - L_d C$. Let $d_f(k) = C\tilde{z}_d(k) = C\hat{z}_d(k) - e(k)$ (measurable).

Let $H_d(z) = C(zI - A_d)^{-1} B$, then $d_f(k) = H_d(z)\,[d(k)]$.

Steady-State Matching Lemma: For the Nikiforov exosystem and disturbance observer, $\exists z_d(0)$ such that the steady-state $d_f(k)$ from the LTI system $z_d(k+1) = (A - L_d C) z_d(k) + Bd(k)$, $d_f(k) = Cz_d(k)$ can be generated by $w_f(k+1) = Fw_f(k) + Gd_f(k)$, $d_f(k) = \psi^T w_f(k)$, and there exists a nonsingular $H_f \in \mathbb{R}^{q \times q}$ such that $w(k) = H_f w_f(k)$.

Intuitively, because $d_f(k)$ is $d(k)$ filtered through an LTI system, it can be generated by a system with the same (F, G, ψ) as the system generating $d(k)$, with the states differing by only a coordinate transformation.

$\boldsymbol{w_f}$ Observer: $\hat{w}_f(k+1) = F\hat{w}_f(k) + Gd_f(k) = F\hat{w}_f(k) + G(C\hat{z}_d(k) - e(k))$

Effective Disturbance:

$$d(k) = \Gamma\xi(k) = \psi^T w(k) = \psi^T H_f^{-1} w_f(k) = \psi_f^T w_f(k) = \psi_f^T \hat{w}_f(k) + \varepsilon(k)$$

Kreisselmeier Filter: Directly estimate $w(k)$:

$$\eta_0(k+1) = F\eta_0(k) + FGe(k) \qquad \eta_2(k+1) = F\eta_2(k) + Gu(k)$$
$$\eta_1(k+1) = F\eta_1(k) - Ge(k) \qquad \hat{w}(k) = \eta_0(k) + Ge(k) - A\eta_1(k) + B\eta_2(k)$$

Controller: $u(k) = u_s(k) + u_{im}(k) = u_s(k) + \hat{\psi}_f^T(k) \hat{w}_f(k)$

Stabilizer: $u_s(k) = K\hat{z}_s(k)$, $\hat{z}_s(k+1) = A\hat{z}_s(k) + Bu_s(k) + L_s(e(k) - C\hat{z}_s(k))$ where $A + BK$ and $A - L_s C$ Schur stable. If A Schur stable, can use $u_s(k) \equiv 0$.

Adaptation: Let $\hat{w}_a(k) = H_d(z) I\left[\hat{w}_f(k)\right]$, then $d_f(k) = \psi_f^T(k) \hat{w}_a(k)$ (swapping lemma). Let $\hat{d}_f(k) = \hat{\psi}_f^T \hat{w}_a(k)$ (measurable) and augmented error $e_a(k) = \hat{d}_f(k) - d_f(k) = \tilde{\psi}^T \hat{w}_a(k)$. Gradient law:

$$\hat{\psi}_f(k+1) = \hat{\psi}_f(k) - \gamma(k) e_a(k) w_a(k) \qquad \gamma(k) = \frac{\bar{\gamma}}{1 + \|w_a(k)\|^2}, \bar{\gamma} \in (0, 2)$$

If $u_s(k) \equiv 0$ (A) Schur stable, can also use dynamic EM for the $z(k)$ system.

9 Useful Identities

$$r(k) = k \implies w(k+1) = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 2 \end{smallmatrix}\right] w(k) \quad r(k) = \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right] w(k) \quad w(0) = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$$

$$r(k) = k^2 \implies w(k+1) = \left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & -3 \end{smallmatrix}\right] w(k) \quad r(k) = \left[\begin{smallmatrix} 1 & 0 & 0 \end{smallmatrix}\right] w(k) \quad w(0) = \left[\begin{smallmatrix} 0 \\ 1 \\ 4 \end{smallmatrix}\right]$$

$$r(k) = \begin{bmatrix} \sin(\omega k) \\ \cos(\omega k) \end{bmatrix} \implies w(k+1) = \begin{bmatrix} \cos(\omega) & \sin(\omega) \\ -\sin(\omega) & \cos(\omega) \end{bmatrix} w(k) \quad r(k) = w(k)$$

$$\sin(\tan^{-1} x) = \frac{x}{\sqrt{1 + x^2}} \qquad \cos(\tan^{-1} x) = \frac{1}{\sqrt{1 + x^2}}$$

$$\sin(\cos^{-1} x) = \cos(\sin^{-1} x) = \sqrt{1 - x^2}$$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2} \qquad \cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$$A^{-1} = \frac{1}{\det A} (\operatorname{cof} A)^T \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\mathcal{Z}\{1\} = \frac{z}{z - 1} \quad \mathcal{Z}\{\delta(k)\} = 1 \quad \mathcal{Z}\{k\} = \frac{z}{(z-1)^2} \quad \sum_{k=0}^{\infty} a x^k = \frac{a}{1 - x}$$