Tutorial 2, Sep 22, 2025

- Show that $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is diagonalizable if and only if it has n linearly independent eigenvectors
  - We have shown that if $\boldsymbol{A}$ has n linearly independent eigenvectors, it is diagonalizable; now we need to show implication in the other direction
  - If $\boldsymbol{A}$ is diagonalizable, by definition there exists $\boldsymbol{P} \in \mathbb{C}^{n\times n}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{\Lambda} \in \mathbb{C}^{n\times n}$ is diagonal
  - Therefore $\boldsymbol{A}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{\Lambda}$
  - $\boldsymbol{A}\boldsymbol{P} = \boldsymbol{A}\begin{bmatrix}\boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n\end{bmatrix} = \begin{bmatrix}\boldsymbol{A}\boldsymbol{v}_1 & \cdots & \boldsymbol{A}\boldsymbol{v}_n\end{bmatrix}$
  - $\boldsymbol{P}\boldsymbol{\Lambda} = \begin{bmatrix}\boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n\end{bmatrix}\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix} = \begin{bmatrix}\lambda_1\boldsymbol{v}_1 & \cdots & \lambda_n\boldsymbol{v}_n\end{bmatrix}$
  - Equating the columns we have that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are the eigenvectors of $\boldsymbol{A}$
  - Now to prove linear independence, we know $\boldsymbol{P}\boldsymbol{v} = \boldsymbol{0} \iff \boldsymbol{v} = \boldsymbol{0}$ as $\boldsymbol{P}$ is invertible; so the only linear combination of the columns of $\boldsymbol{P}$ that results in 0 is all zeros, therefore its columns are linearly independent
- $\boldsymbol{A} = \begin{bmatrix}1 & -1 & -3 & 0\\ 0 & -2 & 0 & 0\\ -3 & -1 & 1 & 0\\ 0 & 0 & 0 & 4\end{bmatrix}$
  - To make the computation for eigenvalues tractable, we expand the determinant along rows that are mostly zeros
  - $$\begin{aligned}\det(\lambda\boldsymbol{I} - \boldsymbol{A}) &= \det\begin{bmatrix}\lambda-1 & -1 & -3 & 0\\ 0 & \lambda+2 & 0 & 0\\ -3 & -1 & \lambda-1 & 0\\ 0 & 0 & 0 & \lambda-4\end{bmatrix}\\ &= (\lambda-4)\det\begin{bmatrix}\lambda-1 & 1 & 3\\ 0 & \lambda+2 & 0\\ 3 & 1 & \lambda-1\end{bmatrix}\\ &= (\lambda-4)(\lambda+2)((\lambda-1)^2-9)\\ &= (\lambda-4)(\lambda+2)(\lambda-4)(\lambda+2)\\ &= (\lambda-4)^2(\lambda+2)^2\end{aligned}$$
  - By inspection the eigenvectors for $\lambda = 4$ are $\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$
  - The only eigenvector for $\lambda = -2$ is $\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$ (we can show that $\operatorname{rank}(-2\boldsymbol{I} - \boldsymbol{A}) = 3$)
  - Since we don't have enough eigenvectors, the matrix is not diagonalizable (using the result from the first question)