Lecture 9, Sep 24, 2025

Stability

Definition

A system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \boldsymbol{x}(0) = \boldsymbol{x}_0$ is said to be *stable* if for every $\boldsymbol{x}_0 \in \mathbb{R}^n$, the solution $\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0$ is *bounded*, i.e.

$$\exists M < \infty \text{ s.t. } \|\boldsymbol{x}(t)\| \leq M, \forall t \geq 0$$

The system is *asymptotically stable* if for every $\boldsymbol{x}_0 \in \mathbb{R}^n$,

$$\lim_{t \to \infty} x(t) = \boldsymbol{0} \in \mathbb{R}^n$$

Theorem

The system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \boldsymbol{x}(0) = \boldsymbol{x}_0$ is asymptotically stable if and only if $\mathrm{Re}(\lambda_i) < 0$ for all eigenvalues λ_i of $\boldsymbol{A}$.

- The intuition here is that we can decouple the system using the eigenvectors like we showed before, and if all eigenvalues have negative real parts, then all components must decay to 0
- Asymptotic stability is equivalent to $e^{\boldsymbol{A}t}$ converging to 0 as $t \to \infty$ (since this is the unique solution)
  - Recall that using the Jordan form, this is equivalent to $e^{\boldsymbol{J}_{\lambda_i} t}$ converging to 0 for each i, which can be expanded to $e^{\lambda_i t}\boldsymbol{N}$ where $\boldsymbol{N}$ is a matrix of polynomials of t
  - Suppose all $\mathrm{Re}(\lambda_i) < 0$; then $e^{\lambda_i t}$ times any polynomial of t will decay to 0 as $t \to \infty$ for all i, since the exponential grows faster than any polynomial
  - Therefore every term in every Jordan block will converge to 0, and so $e^{\boldsymbol{A}t}$ converges to 0 and the system is asymptotically stable
- We can also define some notions of stability when the input $\boldsymbol{u}$ is involved:

Definition

The system

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \end{aligned}$$

is *bounded-input-bounded-output stable* (*BIBO stable*) if, when $\boldsymbol{x}_0 = \boldsymbol{0}$, as long as the input $\boldsymbol{u}(t)$ is bounded, the output $\boldsymbol{y}(t)$ is bounded.

The system

$$\boldsymbol{y}(t) = \int_0^t \boldsymbol{h}(t-\tau)\boldsymbol{u}(\tau)\,\mathrm{d}\tau$$

where $\boldsymbol{h} : [0, \infty) \mapsto \mathbb{R}^{m \times p}$ is BIBO stable if for all bounded $\boldsymbol{u}(t)$, $\boldsymbol{y}(t)$ is also bounded. Note this model implicitly assumes zero inital conditions.

Definition

The system

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \end{aligned}$$

is *input-output stable* if **for all inital conditions** $\boldsymbol{x}_0 \in \mathbb{R}^n$, a bounded $\boldsymbol{u}(t)$ implies a bounded $\boldsymbol{y}(t)$. Note input-output stability implies BIBO stability.

Theorem

If $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ is asymptotically stable, then for any $\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$, the system

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \end{aligned}$$

is both BIBO and input-output stable.

- Recall that PSD for a real symmetric matrix means $\boldsymbol{v}^T\boldsymbol{P}\boldsymbol{v} \geq 0$ for all $\boldsymbol{v} \in \mathbb{R}^n$, and positive definite means $\boldsymbol{v}^T\boldsymbol{P}\boldsymbol{v} > 0$ for all nonzero $\boldsymbol{v}$
  - PSD is sometimes denoted $\boldsymbol{P} \in \mathbb{S}_n^+$; positive definite is denoted $\boldsymbol{P} \in \mathbb{S}_n$
- For complex $\boldsymbol{v}$, $\boldsymbol{v}^*\boldsymbol{P}\boldsymbol{v} > 0$ for positive definite $\boldsymbol{P}$, and greater than or equal to zero for PSD $\boldsymbol{P}$ ($\boldsymbol{v}^*$ denotes a conjugate-transpose or *Hermitian transpose*)
  - $$\begin{aligned} \boldsymbol{v}^*\boldsymbol{P}\boldsymbol{v} &= (\boldsymbol{x} + i\boldsymbol{y})^*\boldsymbol{P}(\boldsymbol{x} + i\boldsymbol{y}) \\ &= (\boldsymbol{x}^T - i\boldsymbol{y}^T)\boldsymbol{P}(\boldsymbol{x} + i\boldsymbol{y}) \\ &= \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} - i\boldsymbol{y}^T\boldsymbol{P}\boldsymbol{x} + i\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{y} - i^2\boldsymbol{y}^T\boldsymbol{P}\boldsymbol{y} \\ &= \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} + \boldsymbol{y}^T\boldsymbol{P}\boldsymbol{y} \end{aligned}$$
    - Since $\boldsymbol{P}$ is positive definite (or PSD) the last two remaining terms are both positive (or nonnegative for PSD)
    - Note that $\boldsymbol{y}^T\boldsymbol{P}\boldsymbol{x}$ is a scalar, so we can take its transpose, and since $\boldsymbol{P}$ is symmetric we can show the expression is equal to $\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{y}$ and so the middle terms cancel

Theorem

Let $\boldsymbol{P} \in \mathbb{R}^{n\times n}$ be symmetric; then $\boldsymbol{P}$ is positive definite if and only if all its eigenvalues are positive; $\boldsymbol{P}$ is positive semidefinite if and only if all its eigenvalues are non-negative. This result is sometimes known as the *spectral theorem*.
Note since $\boldsymbol{P}$ is real and symmetric, all eigenvalues are real.

This theorem also applies for negative (semi-)definite matrices and negative (nonpositive) eigenvalues.

Theorem

Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and suppose there exists a symmetric positive definite matrix $\boldsymbol{P}$ such that

$$\boldsymbol{Q} = -\boldsymbol{A}^T\boldsymbol{P} - \boldsymbol{P}\boldsymbol{A}$$

is also positive definite, then $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ is asymptotically stable. $\boldsymbol{Q}$ is known as the *continuous-time Lyapunov operator*.

- Let $\boldsymbol{e} \in \mathbb{C}^n$ be an eigenvector of $\boldsymbol{A}$; we know $\boldsymbol{e}^*\boldsymbol{Q}\boldsymbol{e} = \boldsymbol{e}^*(-\boldsymbol{A}^T\boldsymbol{P} - \boldsymbol{P}\boldsymbol{A})\boldsymbol{e} > 0$ and we can expand the right hand side, using $\boldsymbol{A}\boldsymbol{e} = \lambda\boldsymbol{e}$, to show that all the eigenvalues of $\boldsymbol{A}$ have negative real parts
- In discrete time, the analogous equation is $\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k$, which turns out to be asymptotically stable if and only if $|\lambda_i| < 1$ for all eigenvalues of $\boldsymbol{A}$
  - The analogous definition of $\boldsymbol{Q}$ is $\mathcal{L}_d(\boldsymbol{P}) = \boldsymbol{P} - \boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A}$ and it also holds that if there exists a $\boldsymbol{P}$ that makes this positive definite, then $\boldsymbol{A}$ has all eigenvalues with magnitude less than 1 and therefore the system is stable
- Now consider $\boldsymbol{A}$ with eigenvalues less than or equal to zero; if we take its Jordan form and expand $e^{\boldsymbol{J}t}$, we find that in some Jordan blocks we only have $e^{\lambda t}$, but in other blocks we have terms with $e^{\lambda t}$ times a polynomial of t
  - We allow the blocks that only have $e^{\lambda t}$ to have a zero eigenvalue, since this becomes a constant
  - However the blocks containing $e^{\lambda t}$ times a polynomial must have a negative eigenvalue, because

  otherwise that term will grow to infinity as $t \to \infty$
  - This means for all Jordan (sub-)blocks that are bigger than 1×1, its λ value must be strictly negative
  - Recall that we get bigger Jordan blocks when the algebraic multiplicity is greater than the geometric multiplicity for some eigenvalue
  - This is the intuition for the next theorem

Theorem

$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ is stable if and only if $\mathrm{Re}(\lambda_i) \leq 0$ for all i, and if for all eigenvalues that have $\mathrm{Re}(\lambda_i) = 0$, the algebraic multiplicity equals the geometric multiplicity for that eigenvalue.

- To prove this, we equate stability to $e^{\boldsymbol{A}t}$ being bounded, which is equal to $e^{\boldsymbol{J}t}$ being bounded, which we can consider separately for negative and zero eigenvalues:
  - $\mathrm{Re}(\lambda_j) < 0$: then all $\lim\limits_{t \to \infty} e^{\boldsymbol{J}_{\lambda_j} t} = 0$ since the exponential grows faster than all polynomials
  - $\mathrm{Re}(\lambda_j) = 0$: $e^{\boldsymbol{J}_{\lambda_j} t}$ is bounded if and only if it has no polynomials in t; this only happens if we have Jordan blocks of size 1, which happens if and only if the geometric and algebraic multiplicities are equal
- Example: $\boldsymbol{A} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$; under what conditions is $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ stable? Asymptotically stable?
  - For both stability and asymptotic stability, we require $\lambda < 0$, since for this matrix λ has algebraic multiplicity of 2 but geometric multiplicity of 1
  - We can see this from $e^{\boldsymbol{A}t} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$ since the matrix is already in Jordan form

Theorem

If $\mathrm{Re}(\lambda_i) > 0$ for any eigenvalue λ_i of $\boldsymbol{A}$, then $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ is unstable.