Lecture 8, Sep 24, 2025

Solution to a Non-Autonomous LTI System

Theorem

The solution to the LTI system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \boldsymbol{x}(0) = \boldsymbol{x}_0$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}$$

is given by

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0 + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau$$

- We can show the initial condition is satisfied trivially
- $\dot{\boldsymbol{x}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\boldsymbol{A}t}\boldsymbol{x}_0 + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau\right)$
  $= \boldsymbol{A}e^{\boldsymbol{A}t}\boldsymbol{x}_0 + \boldsymbol{A}\int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau + \boldsymbol{B}\boldsymbol{u}(t)$
  $= \boldsymbol{A}\left(e^{\boldsymbol{A}t}\boldsymbol{x}_0 + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau\right) + \boldsymbol{B}\boldsymbol{u}(t)$
  $= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$
  - Note $\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau = \boldsymbol{A}\int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau + \boldsymbol{B}\boldsymbol{u}(t)$ by Leibniz rule
    * $\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)} f(x,t)\,\mathrm{d}t = f(x,b(x))\frac{\mathrm{d}}{\mathrm{d}x}b(x) - f(x,a(x))\frac{\mathrm{d}}{\mathrm{d}x}a(x) + \int_{a(x)}^{b(x)}\frac{\partial}{\partial x}f(x,t)\,\mathrm{d}t$
- Note due to the fundamental theorem of differential equations (the existence and uniqueness theorem), as our system is linear (and therefore continuous), we know that the solution above is the unique solution