Lecture 7, Sep 19, 2025

Reasoning About System Behaviour With Eigenvalues and Eigenvectors

- Consider a system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \boldsymbol{x}(0) = \boldsymbol{x}_0$
  - Assume that $\boldsymbol{A}$ is diagonalizable, so the solution is $\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0 = \boldsymbol{P}e^{\boldsymbol{\Lambda}t}\boldsymbol{P}^{-1}\boldsymbol{x}_0$
- Consider the transformed coordinate space $\boldsymbol{z}(t) = \boldsymbol{P}^{-1}\boldsymbol{x}(t)$; how does the system look in this coordinate system?
  - $\dot{\boldsymbol{z}} = \boldsymbol{P}^{-1}\dot{\boldsymbol{x}}(t) = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{x}(t) = \boldsymbol{P}^{-1}\boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}\boldsymbol{x}(t) = \boldsymbol{\Lambda}\boldsymbol{z}(t)$
  - Since $\boldsymbol{\Lambda}$ is diagonal, we get $\dot{z}_i(t) = \lambda_i z_i(t)$, in other words, a set of n decoupled linear differential equations
  - Each one is solved by $z_i(t) = e^{\lambda_i t}z_i(0)$, resulting in much easier to analyze system behaviour
  - $\boldsymbol{x}(t) = \boldsymbol{P}\boldsymbol{z}(t) = \displaystyle\sum_{i=1}^{n}\boldsymbol{v}_i\boldsymbol{z}_i(t) = \sum_{i=1}^{n}\boldsymbol{v}_i e^{\lambda_i t}z_i(0)$
    - Each term of the sum is called the i-th *mode* of $x(t)$; the entire operation is known as a *modal decomposition*
    - We denote $\boldsymbol{h}_i(t) = \displaystyle\sum_{i=1}^{n}\boldsymbol{v}_i e^{\lambda_i t}z_i(0)$
- Geometrically, we can imagine drawing each of the $\boldsymbol{v}_i$ as a line; if the associated λ_i is negative, solutions shrink and go towards 0 along this line; conversely if λ_i is positive, solutions expand and go to infinity along the line
  - For any initial condition $\boldsymbol{x}_0$ we can decompose it into components along each $\boldsymbol{v}_i$, and each of those components will evolve according to λ_i (towards or away from the origin at a speed determined by the magnitude)
  - In the z coordinate system this is easier to see since the $\boldsymbol{v}_i$ are now along the coordinate axes

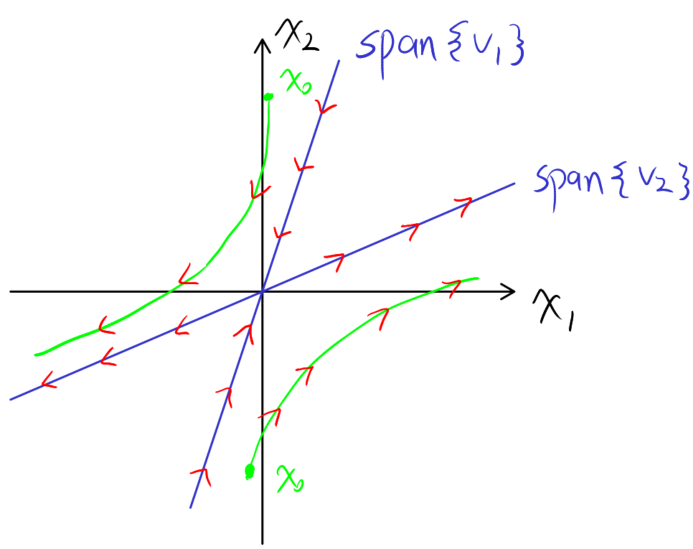

Figure 1: Illustrations of how solutions evolve in the x coordinate system, for an example where $\lambda_1 < 0, \lambda_2 > 0$.

System Behaviour According to Eigenvalues

- With the above knowledge we can now categorize systems according to their eigenvalues
- Case 1: Real and nonzero eigenvalues
  - If all eigenvalues are less than 0, we have a *stable node* since all initial conditions converge towards zero
  - If all eigenvalues are greater than 0, we have an *unstable node* since all initial conditions explode to infinity (except for 0, which stays at 0)

- If eigenvalues have mixed signs, we get a *saddle point* as initial conditions will move towards zero along one axis but diverge away from it on another axis; again, zero is the only initial condition that does not diverge

- Case 2: Complex conjugate eigenvalues $\lambda_1 = a + ib, \lambda_2 = a - ib$ (recall that the solution in this case is $e^{at}\begin{bmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{bmatrix}$)
  - If $a < 0$, we get a *stable focus* as solutions spiral in towards zero
  - If $a > 0$, we get an *unstable focus* as solutions spiral outwards from zero towards infinity
  - If $a = 0$, we get a *centre* since all solutions stay orbiting the origin in a circle, not converging or diverging
  - In all cases, the magnitude determines the rate of spiral
- Case 3: One nonzero eigenvalue
  - The eigenvector with zero eigenvalue forms a line, where every point on the line is an equilibrium
  - If the other eigenvalue is less than zero, all solutions converge towards that line; if the other eigenvalue is greater than zero then all solutions diverge from the line
  - All initial conditions follow a straight path towards the equilibrium line, defined by the other eigenvector (nonzero eigenvalue)

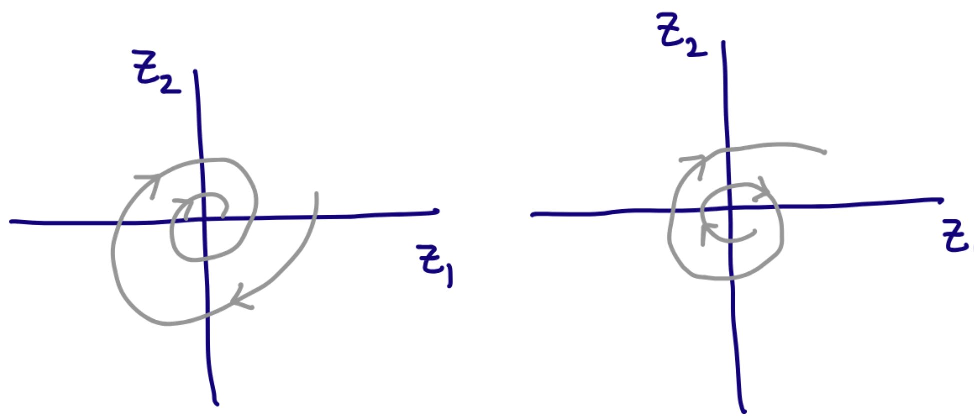

Figure 2: Behaviour for complex eigenvalues.