Lecture 4, Sep 10, 2025

The Matrix Exponential

Definition

Let $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, then the *matrix exponential* is defined as:

$$e^{\boldsymbol{A}} = \sum_{k=0}^{\infty} \frac{1}{k!} \boldsymbol{A}^k$$

Note $\boldsymbol{A}^0 = \boldsymbol{I}_n$.

- To define the matrix exponential based on a series, we have to first define convergence for matrices, and then show that this series definition of the matrix exponential converges

Definition

A series of matrices $\sum_{k=0}^{\infty} \boldsymbol{M}_k$ *converges* if every element $(\boldsymbol{S}_n)_{ij}$ of the partial sum $\boldsymbol{S}_n = \sum_{k=0}^{n} \boldsymbol{M}_k$ converges to a number as $n \to \infty$, i.e. $\lim_{n \to \infty} (\boldsymbol{S}_n)_{ij} = a_{ij}$ for all i, j.

Formally, we require

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n > N \implies |(\boldsymbol{S}_n)_{ij} - a_{ij}| < \epsilon$$

Definition

A *norm* on R^n is a function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ with the following properties:

1. $\|\boldsymbol{x}\| \geq 0 \ \forall x \in \mathbb{R}^n$
2. $\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \boldsymbol{0} \in \mathbb{R}^n$
3. $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\| \ \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$
4. $\|\lambda \boldsymbol{x}\| = |\lambda| \|\boldsymbol{x}\| \ \forall \boldsymbol{x} \in \mathbb{R}^n, \forall \lambda \in \mathbb{R}$

Definition

The *induced norm* on $\mathbb{R}^{n \times n}$ is a function $\|\cdot\| : \mathbb{R}^{n \times n} \to \mathbb{R}$ defined as

$$\|\boldsymbol{A}\| = \max_{\{\, \boldsymbol{x} \in \mathbb{R}^n \,|\, \|\boldsymbol{x}\| = 1 \,\}} \|\boldsymbol{A}\boldsymbol{x}\|$$

i.e. max norm of $\boldsymbol{A}\boldsymbol{x}$ over all $\boldsymbol{x}$ in the unit sphere. Note that this matrix norm is defined in terms of (*induced by*) the vector norm.

- We can show that the induced norm is a valid norm, and it has property $\|\boldsymbol{A}^k\| \leq \|\boldsymbol{A}\|^k$

Theorem

If the scalar series $\sum_{k=0}^{\infty} \|\boldsymbol{M}_k\|$ converges, then the matrix series $\sum_{k=0}^{\infty} \boldsymbol{M}_k$ converges. Such a series is called *absolutely convergent*.

- We will now prove that $e^{\boldsymbol{A}} = \sum_{k=0}^{\infty} \frac{1}{k!} \boldsymbol{A}^k$ is absolutely convergent:

- We need to show that $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=0}^{n} \left\| \frac{\boldsymbol{A}^k}{k!} \right\|$ converges
- We will rely on the fact that an increasing sequence that is bounded above always converges
- S_n is an increasing sequence, since $S_{n+1} - S_n = \left\| \frac{\boldsymbol{A}^{n+1}}{(n+1)!} \right\| = \frac{1}{(n+1)!} \|\boldsymbol{A}^{n+1}\| \geq 0$
- To bound S_n from above, we will show that $S_n \leq e^{\|\boldsymbol{A}\|}$
  * $S_n = \sum_{k=0}^{n} \left\| \frac{\boldsymbol{A}^k}{k!} \right\| \leq \sum_{k=0}^{\infty} \frac{1}{k!} \|\boldsymbol{A}^k\| \leq \sum_{k=0}^{\infty} \frac{1}{k!} \|\boldsymbol{A}\|^k = e^{\|\boldsymbol{A}\|}$

Theorem

The matrix exponential satisfies the following properties:

1. For any invertible $\boldsymbol{P} \in \mathbb{R}^{n\times n}$, $e^{\boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^{-1}} = \boldsymbol{P}e^{\boldsymbol{A}}\boldsymbol{P}^{-1}$
2. For any $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times n}$ such that $AB = BA$ (commutativity) $e^{\boldsymbol{A}+\boldsymbol{B}} = e^{\boldsymbol{A}}e^{\boldsymbol{B}} = e^{\boldsymbol{B}}e^{\boldsymbol{A}}$
3. $(e^{\boldsymbol{A}})^{-1} = e^{-\boldsymbol{A}}$
4. For $t \in \mathbb{R}$, $\frac{\mathrm{d}}{\mathrm{d}t} e^{\boldsymbol{A}t} = \boldsymbol{A}e^{\boldsymbol{A}t} = e^{\boldsymbol{A}t}\boldsymbol{A}$