Lecture 3, Sep 5, 2025

Linearization

- Consider a general nonlinear function $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}), \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u})$ where $\boldsymbol{f}, \boldsymbol{h}$ are differentiable; we want to approximate this system by an LTI model by linearization around an equilibrium point

Definition

A pair $(\boldsymbol{x}^*, \boldsymbol{u}^*)$ is an *equilibrium condition* if $\boldsymbol{f}(\boldsymbol{x}^*, \boldsymbol{u}^*) = \boldsymbol{0}$. In this case $\boldsymbol{x}^*$ is an *equilibrium point* with control $\boldsymbol{u}^*$.

- Consider the example of an actuated pendulum affected by gravity $\boldsymbol{x} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}, y = x_1$
  - $\dot{x}_1 = x_2$
  - $\dot{x}_2 = -\frac{mgl}{J} \sin x_1 + \frac{u}{J}$ (torque balance) where J is the moment of inertia
  - $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) = \begin{bmatrix} x_2 \\ -\frac{mgl}{J} \sin x_1 + \frac{u}{J} \end{bmatrix}$
  - With a control of $u^* = 0$, $\boldsymbol{f}(\boldsymbol{x}^*, 0) = \begin{bmatrix} x_2^* \\ -\frac{mgl}{J} \sin x_1^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \boldsymbol{x}^* = \begin{bmatrix} k\pi \\ 0 \end{bmatrix}, k \in \mathbb{Z}$
    - Physically this corresponds to the pendulum being perfectly up or down with zero velocity
  - With a control of $u^* = mgl$, $\begin{bmatrix} x_2 \\ -\frac{mgl}{J} \sin x_1^* + \frac{mgl}{J} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \boldsymbol{x}^* = \begin{bmatrix} \frac{\pi}{2} + 2k\pi \\ 0 \end{bmatrix}$
    - This corresponds to the applied torque being balanced by gravity
- Consider $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}^*) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}^*)(\boldsymbol{x} - \boldsymbol{x}^*) + \boldsymbol{R}(\boldsymbol{x})$ where $\boldsymbol{R}$ is a remainder term
  - $\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{n_1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{n_2}}{\partial x_1} & \cdots & \frac{\partial f_{n_2}}{\partial n_1} \end{bmatrix} \in \mathbb{R}^{n_2 \times n_1}$ is the *Jacobian* of $\boldsymbol{f}$
  - For differentiable $\boldsymbol{f}$, $\lim_{\boldsymbol{x} \to \boldsymbol{x}^*} \frac{\boldsymbol{R}(\boldsymbol{x})}{\|\boldsymbol{x} - \boldsymbol{x}^*\|} = \boldsymbol{0}$
- Let $\boldsymbol{z}^* = (\boldsymbol{x}^*, \boldsymbol{u}^*), \boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{u})$, then $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{f}(\boldsymbol{z}^*) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{z}}(\boldsymbol{z}^*)(\boldsymbol{z} - \boldsymbol{z}^*) + \boldsymbol{R}(\boldsymbol{z})$ where $\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{z}} = \begin{bmatrix} \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} & \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{u}} \end{bmatrix}$
  - Therefore $\dot{\boldsymbol{x}} \approx \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{z}}(\boldsymbol{z}^*)(\boldsymbol{z} - \boldsymbol{z}^*) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}^*, \boldsymbol{u}^*)(\boldsymbol{x} - \boldsymbol{x}^*) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{x}^*, \boldsymbol{u}^*)(\boldsymbol{u} - \boldsymbol{u}^*)$
    - Note we used the fact that $\boldsymbol{z}^*$ is an equilibrium condition
  - Let $\delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}^*, \delta\boldsymbol{u} = \boldsymbol{u} - \boldsymbol{u}^*$ then $\dot{\delta\boldsymbol{x}} = \dot{\boldsymbol{x}} \approx \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}^*, \boldsymbol{u}^*)\delta\boldsymbol{x} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{x}^*, \boldsymbol{u}^*)\delta\boldsymbol{u}$
  - Similarly let $\delta\boldsymbol{y} = \boldsymbol{y} - \boldsymbol{h}(\boldsymbol{x}^*, \boldsymbol{u}^*)$ then $\delta\boldsymbol{y} \approx \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}(\boldsymbol{x}^*, \boldsymbol{u}^*)\delta\boldsymbol{x} + \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{u}}(\boldsymbol{x}^*, \boldsymbol{u}^*)\delta\boldsymbol{u}$
  - Therefore: $\boldsymbol{A} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}^*, \boldsymbol{u}^*), \boldsymbol{B} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{x}^*, \boldsymbol{u}^*), \boldsymbol{C} = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}(\boldsymbol{x}^*, \boldsymbol{u}^*), \boldsymbol{D} = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{u}}(\boldsymbol{x}^*, \boldsymbol{u}^*)$

Summary

To linearize a general nonlinear system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}), \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u})$ where $\boldsymbol{f}, \boldsymbol{h}$ are differentiable, let $(\boldsymbol{x}^*, \boldsymbol{u}^*)$ be an equilibrium condition, then a linear approximation is

$$\begin{aligned} \dot{\delta \boldsymbol{x}} &= \boldsymbol{A}\delta\boldsymbol{x} + \boldsymbol{B}\delta\boldsymbol{u} \\ \boldsymbol{y} &= \boldsymbol{C}\delta\boldsymbol{x} + \boldsymbol{D}\delta\boldsymbol{u} \end{aligned}$$

where $\delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}^*, \delta\boldsymbol{u} = \boldsymbol{u} - \boldsymbol{u}^*$ and

$$\boldsymbol{A} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}^*, \boldsymbol{u}^*), \boldsymbol{B} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{x}^*, \boldsymbol{u}^*), \boldsymbol{C} = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}(\boldsymbol{x}^*, \boldsymbol{u}^*), \boldsymbol{D} = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{u}}(\boldsymbol{x}^*, \boldsymbol{u}^*)$$