Lecture 2, Sep 5, 2025

Converting Between State Space and Transfer Functions

- Recall the transfer function representation: $Y(s) = G(s)U(s)$ where $U(s) = \mathcal{L}\{u(t)\}$ (input), $Y(s) = \mathcal{L}\{y(t)\}$ (output), the transfer function is $G(s) = \mathcal{L}\{g(t)\}$ (impulse response)
  - Also known as the *input-output representation*
  - Note this assumes zero initial conditions
- Using the circuit example: $\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{1}{R\tilde{C}}y = \frac{1}{R\tilde{C}}u$
  - Assuming zero initial conditions, $\mathcal{L}\{y\}$ and $\mathcal{L}\{u\}$ exist in the right-half complex plane
  - Apply Laplace: $sY(s) + \frac{1}{R\tilde{C}}Y(s) = \frac{1}{R\tilde{C}}U(s) \implies \left(s + \frac{1}{R\tilde{C}}\right)Y(s) = \frac{1}{R\tilde{C}}U(s)$
  - Therefore $G(s) = \frac{\frac{1}{R\tilde{C}}}{s + \frac{1}{R\tilde{C}}}$
  - To go from state space to transfer function representation, we can take the Laplace transform and rearrange into the $Y(s) = G(s)U(s)$ form
- To transform transfer function to state space: Let $G(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{N(s)}{D(s)}$ and assume $a_i, b_i \in \mathbb{R}$ (*rational*) and $m < n$ (*strictly proper*)
  - Break into 2 blocks, $\frac{1}{D(s)}$ and then $N(s)$, and let the intermediate output be $V(s)$; the first block will give us our state equation, the second will give the measurement equation
  - Block 1: $\frac{V(s)}{U(s)} = \frac{1}{D(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$
    - * $\left(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0\right)V(s) = U(s)$
    - * Inverse Laplace assuming zero initial conditions: $\frac{\mathrm{d}^n v}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1}v}{\mathrm{d}t^{n-1}} + \cdots + a_1\frac{\mathrm{d}v}{\mathrm{d}t} + a_0 v = u$
    - * Let $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} v \\ \frac{\mathrm{d}v}{\mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}^{n-1}v}{\mathrm{d}t^{n-1}} \end{bmatrix} \implies \dot{\boldsymbol{x}} = \begin{bmatrix} \frac{\mathrm{d}v}{\mathrm{d}t} \\ \frac{\mathrm{d}^2 v}{\mathrm{d}t^2} \\ \vdots \\ \frac{\mathrm{d}^n v}{\mathrm{d}t^n} \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ -a_{n-1}x_n - \cdots - a_1 x_2 - a_0 x_1 + u \end{bmatrix}$
    - * $\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$
  - Block 2: $Y(s) = V(s)N(s) = (b_m s^m + \cdots + b_1 s + b_0)V(s)$
    - * Again inverse Laplace assuming zero IC ($v(0) = \dot{v}(0) = \cdots = \frac{\mathrm{d}^{m-1}v}{\mathrm{d}t^{m-1}}(0) = 0$)
    - * Using the definition of $\boldsymbol{x}$: $y(t) = b_m\frac{\mathrm{d}^m v}{\mathrm{d}t^m} + \cdots + b_1\frac{\mathrm{d}^v t}{\mathrm{d}+^v}b_0 v = b_m x_{m+1} + \cdots + b_1 x_2 + b_0 x_1$
      - Here is where we use the $m < n$ assumption
    - * Therefore: $\boldsymbol{C} = \begin{bmatrix} b_0 & b_1 & \cdots & b_m \end{bmatrix}, \boldsymbol{D} = 0$
  - Note there are many other sets of $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ that satisfy this

Note

Given the state-space representation with $\boldsymbol{x}(0) = \mathbf{0}$, we can show that the corresponding transfer function is

$$\boldsymbol{G}(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}$$

Note $\boldsymbol{G} \in \mathbb{R}^{p \times m}$ is a matrix. This can be derived by taking the Laplace transform, then isolating and substituting $\boldsymbol{X}$.