Lecture 17, Nov 7, 2025

Stabilizability

- Last lecture we showed that we can place the poles of a closed-loop system arbitrarily if it is controllable; what if the system is not controllable? Can we make it stable?

Definition

$(\boldsymbol{A}, \boldsymbol{B})$ is *stabilizable* if there exists some $\boldsymbol{K}$ such that all the eigenvalues of $(\boldsymbol{A} + \boldsymbol{BK})$ have negative real part, i.e. with control law $\boldsymbol{u} = \boldsymbol{Kx}$, the resulting system is asymptotically stable.

- Stabilizability is a weaker condition than controllability, i.e. controllability implies stabilizability, but not the other way around
- Example: $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
  - Notice this is already in the Kalman decomposition form, so we can tell that the system is not controllable
  - This has eigenvalues $\{1, -1\}$, where we cannot affect the -1 since it is in $\hat{\boldsymbol{A}}_{22}$; however we can affect the other eigenvalue of 1 to bring it into the open left half plane
  - Consider $\boldsymbol{u} = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \boldsymbol{x}$, so the closed-loop system is $\boldsymbol{A} + \boldsymbol{bk} = \begin{bmatrix} 1 + k_1 & 1 + k_2 \\ 0 & -1 \end{bmatrix}$
  - Therefore we can choose any k_2, and choose a k_1 such that $k_1 < -1$, so that $1 + k_1$ (the first eigenvalue) has negative real part
  - The speed of convergence is capped by the uncontrollable eigenvalue of -1, so regardless of our choice of k_1, the system cannot possibly converge faster than e^{-t}

Definition

For a system $(\boldsymbol{A}, \boldsymbol{B})$, in its Kalman decomposition, the eigenvalues of $\hat{\boldsymbol{A}}_{22}$ are the *uncontrollable eigenvalues*; the other eigenvalues, i.e. the eigenvalues of $\hat{\boldsymbol{A}}_{11}$, are the *controllable eigenvalues*. Note the eigenvalues of $\boldsymbol{A}$ (equivalently the eigenvalues of $\hat{\boldsymbol{A}}$) is the union of the eigenvalues of $\hat{\boldsymbol{A}}_{11}, \hat{\boldsymbol{A}}_{22}$.

- An equivalent definition for stabilizability is to have all the uncontrollable eigenvalues have negative real part, or equivalently all the nonnegative eigenvalues are controllable

Theorem

PBH Test for Stabilizability: λ is a controllable eigenvalue of $(\boldsymbol{A}, \boldsymbol{B})$ (equivalently, λ is not an eigenvalue of $\hat{\boldsymbol{A}}_{22}$) if and only if

$$\operatorname{rank}(\begin{bmatrix} \lambda \boldsymbol{I} - \boldsymbol{A} & \boldsymbol{B} \end{bmatrix}) = n$$

Equivalently, λ is an uncontrollable eigenvalue if and only if $\operatorname{rank}(\begin{bmatrix} \lambda \boldsymbol{I} - \boldsymbol{A} & \boldsymbol{B} \end{bmatrix}) < n$. Therefore a system is stabilizable if and only if this matrix has rank n for all non-negative eigenvalues of $\boldsymbol{A}$.

- Proof of forward direction (λ not an eigenvalue of $\hat{\boldsymbol{A}}_{22} \implies \operatorname{rank}(\begin{bmatrix} \lambda \boldsymbol{I} - \boldsymbol{A} & \boldsymbol{B} \end{bmatrix}) = n$):
  - $\operatorname{rank}(\begin{bmatrix} s\boldsymbol{I} - \hat{\boldsymbol{A}} & \hat{\boldsymbol{B}} \end{bmatrix}) = \operatorname{rank}(\begin{bmatrix} s\boldsymbol{I} - \boldsymbol{A} & \boldsymbol{B} \end{bmatrix})$ because the two matrices are related through a matrix multiplication by a non-singular matrix
  - $\begin{bmatrix} \lambda \boldsymbol{I} - \hat{\boldsymbol{A}} & \hat{\boldsymbol{B}} \end{bmatrix} = \begin{bmatrix} \lambda \boldsymbol{I} - \hat{\boldsymbol{A}}_{11} & -\hat{\boldsymbol{A}}_{12} & \hat{\boldsymbol{B}}_1 \\ 0 & \lambda \boldsymbol{I} - \hat{\boldsymbol{A}}_{22} & 0 \end{bmatrix}$
  - If λ is not an eigenvalue of $\hat{\boldsymbol{A}}_{22}$, then $\lambda \boldsymbol{I} - \hat{\boldsymbol{A}}_{22}$ is full-rank and therefore the bottom $n - k$ rows are linearly independent, so we only need to look at the top k rows (where $k = \operatorname{rank}(\boldsymbol{Q}_c)$)
  - We showed in lecture that the subsystem $(\hat{\boldsymbol{A}}_{11}, \hat{\boldsymbol{B}}_1)$ is completely controllable, and therefore $\operatorname{rank}(\begin{bmatrix} \lambda \boldsymbol{I} - \hat{\boldsymbol{A}}_{11} & \hat{\boldsymbol{B}}_1 \end{bmatrix}) = k$ by the PBH controllability test

  - Since adding the extra columns in $-\hat{\boldsymbol{A}}_{12}$ cannot possibly make the first k rows dependent, we conclude that the first k rows are linearly independent, so the overall matrix has rank n
- In general, if we have a system that is stabilizable but not controllable, we can find its Kalman decomposition, and design a controller to stabilize the controllable subsystem only, and then transform back
  - For the uncontrollable subsystem the gain would be arbitrary, so we usually just append zeros