Lecture 16, Nov 7, 2025

Pole Assignment for Multi-Input Systems

- Proof of Wonham's pole assignment theorem:
  - Assume $(\boldsymbol{A}, \boldsymbol{B})$ controllable, show that we can assign the poles to $\Lambda = \{ \lambda_1, \dots, \lambda_n \}$
    * Let some nonzero $\boldsymbol{b} = \boldsymbol{B}\boldsymbol{g}$, then by the helper lemma from the previous lecture, there exists an $\boldsymbol{F}$ such that $(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}, \boldsymbol{b})$ is controllable
    * Since this is a single-input system and controllable, we know pole assignment is solvable, i.e. $\exists \boldsymbol{H} \in \mathbb{R}^{1 \times n}$ such that $(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F} + \boldsymbol{b}\boldsymbol{H})$ has eigenvalues Λ
    * $$\begin{aligned} \boldsymbol{A} + \boldsymbol{B}\boldsymbol{F} + \boldsymbol{b}\boldsymbol{H} &= \boldsymbol{A} + \boldsymbol{B}\boldsymbol{F} + \boldsymbol{B}\boldsymbol{g}\boldsymbol{H} \\ &= \boldsymbol{A} + \boldsymbol{B}(\boldsymbol{F} + \boldsymbol{g}\boldsymbol{H}) \\ &= \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K} \end{aligned}$$
    * Therefore if we choose $\boldsymbol{K} = \boldsymbol{F} + \boldsymbol{g}\boldsymbol{H}$, then the closed-loop system will have eigenvalues Λ
  - To show the reverse direction, take the contrapositive, i.e. assume that $(\boldsymbol{A}, \boldsymbol{B})$ is not controllable, show that the pole assignment problem is not solvable
    * Use the Kalman decomposition $\begin{bmatrix} \dot{\boldsymbol{z}}^1 \\ \dot{\boldsymbol{z}}^2 \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{A}}_{11} & \hat{\boldsymbol{A}}_{12} \\ 0 & \hat{\boldsymbol{A}}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{z}^1 \\ \boldsymbol{z}^2 \end{bmatrix} + \begin{bmatrix} \hat{\boldsymbol{B}}_1 \\ 0 \end{bmatrix} \boldsymbol{u}$
    * Let $\hat{\boldsymbol{K}} = \begin{bmatrix} \hat{\boldsymbol{K}}_1 & \hat{\boldsymbol{K}}_2 \end{bmatrix}$ so that $\boldsymbol{u} = \hat{\boldsymbol{K}} \begin{bmatrix} \boldsymbol{z}^1 \\ \boldsymbol{z}^2 \end{bmatrix} = \hat{\boldsymbol{K}}_1 \boldsymbol{z}^1 + \hat{\boldsymbol{K}}_2 \boldsymbol{z}^2$
    * The closed-loop system is $\begin{bmatrix} \dot{\boldsymbol{z}}^1 \\ \dot{\boldsymbol{z}}^2 \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{A}}_{11} + \hat{\boldsymbol{B}}_1 \hat{\boldsymbol{K}}_1 & \hat{\boldsymbol{A}}_{12} + \hat{\boldsymbol{B}}_1 \hat{\boldsymbol{K}}_2 \\ 0 & \hat{\boldsymbol{A}}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{z}^1 \\ \boldsymbol{z}^2 \end{bmatrix} = \hat{\boldsymbol{M}} \boldsymbol{z}$
    * Since $\hat{\boldsymbol{M}}$ is block-upper-triangular, its eigenvalues are the union of the eigenvalues of $\hat{\boldsymbol{A}}_{11} + \hat{\boldsymbol{B}}_1 \hat{\boldsymbol{K}}_1$ and those of $\hat{\boldsymbol{A}}_{22}$
    * However, we cannot affect the eigenvalues of $\hat{\boldsymbol{A}}_{22}$ through $\hat{\boldsymbol{K}}$ at all, so in general pole assignment is not solvable
- Wonham's pole assignment theorem also leads to an eigenvalue assignment algorithm for multi-input systems, by converting it into a single-input system
  - Like in the proof, we convert the system to a single-input one, $(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{F}, \boldsymbol{b})$, then use the single-input pole assignment algorithm to find $\boldsymbol{H}$, and finally take $\boldsymbol{K} = \boldsymbol{F} + \boldsymbol{g}\boldsymbol{H}$