Lecture 15, Nov 7, 2025

Stabilization and Pole Assignment

- The *stabilization problem* is to design a controller $\boldsymbol{u} = \boldsymbol{K}\boldsymbol{x}$ so that the resulting system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K})\boldsymbol{x}$ is asymptotically stable
  - For a nonlinear system, we want $\boldsymbol{x}(t) \to \boldsymbol{x}^*(t)$ where $\boldsymbol{x}^*(t)$ is an equilibrium condition, in which case $\tilde{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}^* = \boldsymbol{K}(\boldsymbol{x} - \boldsymbol{x}^*)$ and the system is $\dot{\tilde{\boldsymbol{x}}} = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K})\tilde{\boldsymbol{x}}$
- The *pole assignment problem* is to find $\boldsymbol{K}$ such that the eigenvalues of $\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}$ are in designed locations of $\mathbb{C}$

Theorem

If the single-input system $(\boldsymbol{A}, \boldsymbol{b})$ is controllable, then the pole assignment problem is solvable, i.e. the eigenvalues of $(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K})$ can be placed arbitrarily, as long as they are in conjugate pairs.

- Proof: We can put $(\boldsymbol{A}, \boldsymbol{b})$ in controllable canonical form, $\dot{\boldsymbol{z}} = \hat{\boldsymbol{A}}\boldsymbol{z} + \hat{\boldsymbol{b}}\boldsymbol{u}$
  - Let $\hat{\boldsymbol{k}} = \begin{bmatrix} \hat{k}_1 & \cdots & \hat{k}_n \end{bmatrix}$
  - The closed-loop system looks like $\boldsymbol{A} + \hat{\boldsymbol{b}}\hat{\boldsymbol{k}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \hat{k}_1 - a_0 & \hat{k}_2 - a_1 & \hat{k}_3 - a_2 & \cdots & \hat{k}_n - a_{n-1} \end{bmatrix}$
  - We can show $\det(s\boldsymbol{I} - (\boldsymbol{A} + \hat{\boldsymbol{b}}\hat{\boldsymbol{k}})) = s^n + (a_{n-1} - \hat{k}_n)s^{n-1} + \cdots + (a_1 - \hat{k}_2)s + (a_0 - \hat{k}_1)$
  - Therefore if we want some set of poles $\lambda_1, \dots, \lambda_n$, we can expand $(s - \lambda_1)(s - \lambda_2)\cdots(s - \lambda_n) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0$, then take $\hat{k}_i = a_{i-1} - \alpha_{i-1}$
  - To get $\boldsymbol{k}$ for the original system (if it was not originally in controllable canonical form), note $\boldsymbol{k}\boldsymbol{x} = \boldsymbol{u} = \hat{\boldsymbol{k}}\boldsymbol{z} = \hat{\boldsymbol{k}}\boldsymbol{P}^{-1}\boldsymbol{x}$, so $\boldsymbol{k} = \hat{\boldsymbol{k}}\boldsymbol{P}^{-1}$

Theorem

Wonham's Pole Assignment Theorem: Any system $(\boldsymbol{A}, \boldsymbol{B})$ is completely controllable if and only if the poles of $\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}$ can be freely assigned, i.e. the pole assignment problem is solvable.

- Lemma: If $(\boldsymbol{A}, \boldsymbol{B})$ is completely controllable, then $\forall \boldsymbol{b} \in \mathcal{R}(\boldsymbol{B}), \exists \boldsymbol{F} \in \mathbb{R}^{m \times n}$ such that $(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}, \boldsymbol{b})$ is controllable
  - This essentially transforms the multi-input case into the single-input case
  - We will not prove this in lecture, but we'll use this for the proof of Wonham's pole assignment theorem

Summary

To place the closed-loop poles of $(\boldsymbol{A}, \boldsymbol{b})$, i.e. set the eigenvalues of $\boldsymbol{A} + \boldsymbol{kb}$ to $\{\, \lambda_1, \ldots, \lambda_n \,\}$:

1. Expand the desired characteristic polynomial:

$$p_{des}(s) = (s - \lambda_1) \cdots (s - \lambda_n) = s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0$$

2. Expand the characteristic polynomial of $\boldsymbol{A}$:

$$p_A(s) = s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

3. Let $\hat{\boldsymbol{k}} = \begin{bmatrix} a_0 - \alpha_0 & \cdots & a_{n-1} - \alpha_{n-1} \end{bmatrix}$
4. Determine $\boldsymbol{P}$ required to put the system into controllable canonical form:

$$\boldsymbol{P} = \boldsymbol{Q}_c \boldsymbol{T} \qquad \boldsymbol{T} = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

5. Let $\boldsymbol{k} = \hat{\boldsymbol{k}} \boldsymbol{P}^{-1}$ and $\boldsymbol{u} = \boldsymbol{kx}$ solves the pole assignment problem.