# Lecture 14, Oct 17, 2025

## Kalman Decomposition and Controllable Canonical Form

**Theorem**

*PBH Controllability Test*: The system $(\boldsymbol{A}, \boldsymbol{B})$ is completely controllable if and only if

$$\operatorname{rank}(\begin{bmatrix} s\boldsymbol{I} - \boldsymbol{A} & \boldsymbol{B} \end{bmatrix}) = n, \forall s \in \mathbb{C}$$

Note if $s$ is not an eigenvalue of $\boldsymbol{A}$, this matrix always has rank $n$, so the condition only needs to be tested for eigenvalues of $\boldsymbol{A}$.

- Proof:
  - Forward direction: $\operatorname{rank}(\boldsymbol{Q}_c) = n \implies \operatorname{rank}(\begin{bmatrix} s\boldsymbol{I} - \boldsymbol{A} & \boldsymbol{B} \end{bmatrix}) = n, \forall s \in \mathbb{C}$
    - Take the contrapositive, $\exists s \in \mathbb{C}$ s.t. $\operatorname{rank}(\begin{bmatrix} s\boldsymbol{I} - \boldsymbol{A} & \boldsymbol{B} \end{bmatrix}) < n \implies \operatorname{rank}(\boldsymbol{Q}_c) < n$
    - Since the matrix is not full rank, there exists $\boldsymbol{v}$ such that $\boldsymbol{v}^T \begin{bmatrix} s\boldsymbol{I} - \boldsymbol{A} & \boldsymbol{B} \end{bmatrix} = \boldsymbol{0}$
    - Therefore $s\boldsymbol{v}^T = \boldsymbol{v}^T\boldsymbol{A}$ and $\boldsymbol{v}^T\boldsymbol{B} = \boldsymbol{0} \in \mathbb{R}^{1\times m}$
    - Multiply by $\boldsymbol{B}$, $s\boldsymbol{v}^T\boldsymbol{B} = \boldsymbol{v}^T\boldsymbol{A}\boldsymbol{B}$, but $\boldsymbol{v}^T\boldsymbol{B} = 0$ so $\boldsymbol{v}^T\boldsymbol{A}\boldsymbol{B} = 0$
      - We can repeat this for all powers of $\boldsymbol{A}$, e.g. $s\boldsymbol{v}^T\boldsymbol{A}\boldsymbol{B} = \boldsymbol{v}^T\boldsymbol{A}^2\boldsymbol{B} = 0$
    - Therefore $\boldsymbol{v}^T \begin{bmatrix} \boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix} = \boldsymbol{v}^T\boldsymbol{Q}_c = \boldsymbol{0}$, and so $\boldsymbol{Q}_c$ is not full rank
- Note that since multiplying by a non-singular matrix does not change rank, we can show that the PBH test is coordinate invariant
- Suppose $\operatorname{rank}(\boldsymbol{Q}_c) < n$, i.e. $\mathcal{R}(\boldsymbol{Q}_c) \subsetneq \mathbb{R}^n$; $\mathcal{R}(\boldsymbol{Q}_c)$ has the following properties:
  - $\mathcal{R}(\boldsymbol{Q}_c)$ is $\boldsymbol{A}$-invariant
  - $\mathcal{R}(\boldsymbol{B}) \subseteq \mathcal{R}(\boldsymbol{Q}_c)$
- As a consequence of the above and the representation theorem, there exists a nonsingular matrix $\boldsymbol{P}$ such that $\begin{bmatrix} \hat{\boldsymbol{A}}_{11} & \hat{\boldsymbol{A}}_{12} \\ \boldsymbol{0} & \hat{\boldsymbol{A}}_{12} \end{bmatrix} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ and $\begin{bmatrix} \hat{\boldsymbol{B}}_1 \\ \boldsymbol{0} \end{bmatrix} = \boldsymbol{P}^{-1}\boldsymbol{B}$, where $\hat{\boldsymbol{A}}_{11}, \hat{\boldsymbol{B}}_1$ have dimension $\dim(\mathcal{R}(\boldsymbol{Q}_c))$
- Let $\boldsymbol{z} = \boldsymbol{P}^{-1}\boldsymbol{x} = \begin{bmatrix} \boldsymbol{z}^1 \\ \boldsymbol{z}^2 \end{bmatrix}$ where $\boldsymbol{z}^1 \in \mathbb{R}^{\dim(\mathcal{R}(\boldsymbol{Q}_c))}$ and $\boldsymbol{z}^2$ has the dimensions of its independent complement
  - $\dot{\boldsymbol{z}}^1 = \hat{\boldsymbol{A}}_{11}\boldsymbol{z}^1 + \hat{\boldsymbol{A}}_{12}\boldsymbol{z}^2 + \hat{\boldsymbol{B}}_1\boldsymbol{u}$
  - $\dot{\boldsymbol{z}}^2 = \hat{\boldsymbol{A}}_{22}\boldsymbol{z}^2$
  - This is the *Kalman decomposition for controllability*
- The Kalman decomposition separates the system into a part that we can control and a part we cannot, so if the eigenvalues of $\hat{\boldsymbol{A}}_{22}$ don't have negative real parts, our system cannot be controlled

**Definition**

For a system $(\boldsymbol{A}, \boldsymbol{b})$ where $\operatorname{rank}(\boldsymbol{Q}_c) = k < n$, let

$$\begin{bmatrix} \hat{\boldsymbol{A}}_{11} & \hat{\boldsymbol{A}}_{12} \\ \boldsymbol{0} & \hat{\boldsymbol{A}}_{12} \end{bmatrix} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}, \begin{bmatrix} \hat{\boldsymbol{B}}_1 \\ \boldsymbol{0} \end{bmatrix} = \boldsymbol{P}^{-1}\boldsymbol{B}$$

for some nonsingular $\boldsymbol{P}$, where $\hat{\boldsymbol{A}}_{11}, \hat{\boldsymbol{B}}_1 \in \mathbb{R}^{k\times k}$, $\hat{\boldsymbol{A}}_{12} \in \mathbb{R}^{k\times(n-k)}$, $\hat{\boldsymbol{A}}_{22} \in \mathbb{R}^{(n-k)\times(n-k)}$.

The *Kalman decomposition* defines $\boldsymbol{z} = \boldsymbol{P}^{-1}\boldsymbol{x} = \begin{bmatrix} \boldsymbol{z}^1 \\ \boldsymbol{z}^2 \end{bmatrix}^T$ where $\boldsymbol{z}^1 \in \mathbb{R}^k, \boldsymbol{z}^2 \in \mathbb{R}^{n-k}$, so the system is decomposed as

$$\dot{\boldsymbol{z}}^1 = \hat{\boldsymbol{A}}_{11}\boldsymbol{z}^1 + \hat{\boldsymbol{A}}_{12}\boldsymbol{z}^2 + \hat{\boldsymbol{B}}_1\boldsymbol{u}$$
$$\dot{\boldsymbol{z}}^2 = \hat{\boldsymbol{A}}_{22}\boldsymbol{z}^2$$

- We can show that $(\hat{\boldsymbol{A}}_{11}, \hat{\boldsymbol{B}}_1)$ is completely controllable

- $\boldsymbol{P}^{-1}\boldsymbol{Q}_c = \begin{bmatrix} \boldsymbol{P}^{-1}\boldsymbol{B} & \cdots \\ \boldsymbol{P}^{-1}\boldsymbol{A}^{n-1}\boldsymbol{B} & = \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{B}}_1 & \hat{\boldsymbol{A}}_{11}\hat{\boldsymbol{B}}_1 & \cdots & \hat{\boldsymbol{A}}_{11}^{n-1}\hat{\boldsymbol{B}}_1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$
- Note $k = \text{rank}(\boldsymbol{Q}_c)$, and since $\boldsymbol{P}$ is invertible, $\text{rank}(\boldsymbol{P}^{-1}Q_c) = k$
- Since the zeros at the bottom don't affect rank, $\text{rank}(\begin{bmatrix}\hat{\boldsymbol{B}}_1 & \hat{\boldsymbol{A}}_{11}\hat{\boldsymbol{B}}_1 & \cdots & \hat{\boldsymbol{A}}_{11}^{n-1}\hat{\boldsymbol{B}}_1\end{bmatrix}) = k$
  - We're not done yet because we want the last power of $\hat{\boldsymbol{A}}_{11}$ to be $k-1$
- By Cayley-Hamilton, we know $\hat{\boldsymbol{A}}_{11}^{k}, \hat{\boldsymbol{A}}_{11}^{k+1}, \dots, \hat{\boldsymbol{A}}_{11}^{n-1}$ can all be written as a linear combination of $\hat{\boldsymbol{A}}_{11}, \dots, \hat{\boldsymbol{A}}_{11}^{k-1}$, because $\hat{\boldsymbol{A}}_{11} \in \mathbb{R}^{k\times k}$
- This means $\text{rank}(\begin{bmatrix}\hat{\boldsymbol{B}}_1 & \hat{\boldsymbol{A}}_{11}\hat{\boldsymbol{B}}_1 & \cdots & \hat{\boldsymbol{A}}_{11}^{n-1}\hat{\boldsymbol{B}}_1\end{bmatrix}) = \text{rank}(\begin{bmatrix}\hat{\boldsymbol{B}}_1 & \hat{\boldsymbol{A}}_{11}\hat{\boldsymbol{B}}_1 & \cdots & \hat{\boldsymbol{A}}_{11}^{k-1}\hat{\boldsymbol{B}}_1\end{bmatrix}) = k$
  - This is a simple extension of what we proved in Assignment 3

- Because $\hat{\boldsymbol{A}}$ is a block-upper-triangular matrix, its eigenvalues are the union of eigenvalues of $\hat{\boldsymbol{A}}_{11}, \hat{\boldsymbol{A}}_{22}$; furthermore, the similarity transform by $\boldsymbol{P}$ does not affect eigenvalues, so the eigenvalues of $\boldsymbol{A}$ are also this same set
  - We can control the eigenvalues of the $\hat{\boldsymbol{A}}_{11}$ subsystem; these are known as the *controllable modes/eigenvalues*
  - However we can't control the eigenvalues of $\hat{\boldsymbol{A}}_{22}$, so these are the *uncontrollable modes/eigenvalues*
  - Intuitively we can see this because the control $\boldsymbol{u}$ applies to $\hat{\boldsymbol{A}}_{11}$ but not $\hat{\boldsymbol{A}}_{22}$
- Intuition: The rank of the controllability matrix is the number of states that are controllable; therefore if the rank is $n$, then all states are controllable, but if the rank is less than $n$, some states will not be controllable and so it might not be possible to stabilize the system
- Practically, to compute the Kalman decomposition, we need to select a basis for $\mathcal{R}(\boldsymbol{Q}_c)$ (e.g. by picking independent columns), and then select $n-k$ other linearly independent vectors that together form a basis for $\mathbb{R}^n$; then we can form $\boldsymbol{P}$ and compute $\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}$
  - The choice of basis does affect the form of $\hat{\boldsymbol{A}}_{11}$ and $\hat{\boldsymbol{A}}_{22}$, however it does not change the controllable and uncontrollable eigenvalues

### Controllable Canonical Form

- Consider a single input system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u$ where $(\boldsymbol{A}, \boldsymbol{b})$ is completely controllable; by choosing a special basis, we can write this in a standard form known as the *controllable canonical form*
- Let $X_A(s) = \det(s\boldsymbol{I} - \boldsymbol{A}) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$, the characteristic polynomial of $\boldsymbol{A}$
- Define our series of basis vectors:
  - $\boldsymbol{v}^n = \boldsymbol{b}$
  - $\boldsymbol{v}^{n-1} = \boldsymbol{A}\boldsymbol{v}^n + a_{n-1}\boldsymbol{v}^n = \boldsymbol{A}\boldsymbol{b} + a_{n-1}\boldsymbol{b}$
  - $\boldsymbol{v}^{n-2} = \boldsymbol{A}\boldsymbol{v}^{n-1} + a_{n-2}\boldsymbol{v}^n = \boldsymbol{A}^2\boldsymbol{b} + a_{n-1}\boldsymbol{A}\boldsymbol{b} + a_{n-2}\boldsymbol{b}$
  - $\cdots$
  - $\boldsymbol{v}^1 = \boldsymbol{A}\boldsymbol{v}^2 + a_1\boldsymbol{v}^n = \boldsymbol{A}^{n-1}\boldsymbol{b} + a_{n-1}\boldsymbol{A}^{n-2}\boldsymbol{b} + \cdots + a_1\boldsymbol{b}$
- Note that $\boldsymbol{A}\boldsymbol{v}^i = \boldsymbol{v}^{i-1} - a_{i-1}\boldsymbol{v}^n$ and $\boldsymbol{A}\boldsymbol{v}^1 + a_0\boldsymbol{v}^n = 0$
  - By Cayley-Hamilton, $$\begin{aligned} &\boldsymbol{A}(\boldsymbol{A}^{n-1} + a_{n-1}\boldsymbol{A}^{n-2} + \cdots + a_1\boldsymbol{I}) + a_0\boldsymbol{I} = 0 \\ \implies &\boldsymbol{A}(\boldsymbol{A}^{n-1} + a_{n-1}\boldsymbol{A}^{n-2} + \cdots + a_1\boldsymbol{I})\boldsymbol{b} + a_0\boldsymbol{b} = 0 \end{aligned}$$
  - Notice that $$\begin{aligned} \boldsymbol{A}\boldsymbol{v}^1 + a_0\boldsymbol{v}^n &= \boldsymbol{A}(\boldsymbol{A}^{n-1}\boldsymbol{b} + a_{n-1}\boldsymbol{A}^{n-2}\boldsymbol{b} + \cdots + a_1\boldsymbol{b}) + a_0\boldsymbol{b} \\ &= \boldsymbol{A}(\boldsymbol{A}^{n-1} + a_{n-1}\boldsymbol{A}^{n-2} + \cdots + a_1\boldsymbol{I})\boldsymbol{b} + a_0\boldsymbol{b} \\ &= 0 \end{aligned}$$
- To show that $\{\, \boldsymbol{v}^1, \dots, \boldsymbol{v}^n \,\}$ is linearly independent:
  - $$\begin{bmatrix}\boldsymbol{v}^1 & \cdots & \boldsymbol{v}^n\end{bmatrix} = \begin{bmatrix}\boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{b}\end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & \cdots & a_{n-1} & 1 & 0 \\ a_3 & \vdots & \ddots & & & \\ \vdots & a_{n-1} & & & & \\ a_{n-1} & 1 & & & & \\ 1 & 0 & & & & \end{bmatrix} = \boldsymbol{Q}_c\boldsymbol{T}$$
  - Due to this structure, $\det(\boldsymbol{T}) = (-1)^{n-1}$; since the system is controllable, we know $\boldsymbol{Q}_c$ is invertible, and therefore $\begin{bmatrix}v^1 & \cdots & v^n\end{bmatrix}$ is also invertible

- Let $\boldsymbol{P} = \begin{bmatrix} \boldsymbol{v}^1 & \cdots & \boldsymbol{v}^n \end{bmatrix}$ and $\boldsymbol{z} = \boldsymbol{P}^{-1}\boldsymbol{x} \implies \dot{\boldsymbol{z}} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}\boldsymbol{z} + \boldsymbol{P}^{-1}\boldsymbol{b}\boldsymbol{u} = \tilde{\boldsymbol{A}}\boldsymbol{z} + \tilde{\boldsymbol{b}}\boldsymbol{u}$
  - $\tilde{\boldsymbol{b}} = \boldsymbol{P}^{-1}\boldsymbol{b} \implies \boldsymbol{P}\tilde{\boldsymbol{b}} = \boldsymbol{b} = \boldsymbol{v}^n$, so $\tilde{\boldsymbol{b}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$
  - $\tilde{\boldsymbol{A}} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} \implies \boldsymbol{A}\boldsymbol{P} = \boldsymbol{P}\tilde{\boldsymbol{A}}$
  - $\boldsymbol{A}\boldsymbol{P} = \begin{bmatrix} \boldsymbol{A}\boldsymbol{v}^1 & \boldsymbol{A}\boldsymbol{v}^2 & \cdots & \boldsymbol{A}\boldsymbol{v}^n \end{bmatrix}$

    $= \begin{bmatrix} -a_0\boldsymbol{v}^n & \boldsymbol{v}^1 - a_1\boldsymbol{v}^n & \boldsymbol{v}^2 - a_2\boldsymbol{v}^n & \cdots & \boldsymbol{v}^{n-1} - a_{n-1}\boldsymbol{v}^n \end{bmatrix}$

    $= \begin{bmatrix} \boldsymbol{v}^1 & \cdots & \boldsymbol{v}^n \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}$

    $= \boldsymbol{P}\tilde{\boldsymbol{A}}$
- We have just proven that a completely controllable system can be written in controllable canonical form; it turns out that the reverse is also true, i.e. if a system can be written in controllable canonical form, it is always controllable

**Theorem**

A single-input system $(\boldsymbol{A}, \boldsymbol{b})$ is completely controllable if and only if there exists a nonsingular matrix $\boldsymbol{P}$, such that

$$\dot{\boldsymbol{z}} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}\boldsymbol{z} + \boldsymbol{P}^{-1}\boldsymbol{b}\boldsymbol{u} = \tilde{\boldsymbol{A}}\boldsymbol{z} + \tilde{\boldsymbol{b}}\boldsymbol{u}$$

where

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \qquad \tilde{\boldsymbol{b}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

This is known as the *controllable canonical form.*