Lecture 13, Oct 15, 2025

Controllability

Definition

An LTI system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$ is *completely controllable* (or just controllable) if, for some positive time T, for all possible initial and final states $\boldsymbol{x}_0, \boldsymbol{x}_f \in \mathbb{R}^n$, there exists some piecewise continuous input $\boldsymbol{u}(t), t \in [0, T]$ that brings the system from the initial state to the final state, i.e.

$$\boldsymbol{x}_f = \boldsymbol{x}(T) = e^{\boldsymbol{A}T}\boldsymbol{x}_0 + \int_0^T e^{\boldsymbol{A}(T-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau = e^{\boldsymbol{A}\boldsymbol{T}} + L_c(\boldsymbol{u}(\cdot))$$

- $L_c(\boldsymbol{u}(\cdot))$ is a map from real piecewise continuous functions to $\mathbb{R}^n$, the impact of the input on the final state (compared to just an autonomous system)
- Let $\bar{\mathcal{R}}_T(\boldsymbol{x}_0) = \{\, \boldsymbol{x}_f \in \mathbb{R}^n \mid \exists u : [0, T] \mapsto \mathbb{R}^m, \boldsymbol{x}_f = e^{\boldsymbol{A}\boldsymbol{T}}\boldsymbol{x}_0 + L_c(\boldsymbol{u}(\cdot)) \,\}$
  - This denotes the set of all possible final states $\boldsymbol{x}_f$ that we can reach from an initial state $\boldsymbol{x}_0$ with piecewise continuous inputs $\boldsymbol{u}$
- Lemma: The LTI system $(\boldsymbol{A}, \boldsymbol{B})$ is completely controllable if and only if $\bar{\mathcal{R}}_T(0) = \mathbb{R}^n$, or equivalently $\mathcal{R}(L_c) = \mathbb{R}^n$
  - Assume $(\boldsymbol{A}, \boldsymbol{B})$ is completely controllable, then we can let $\boldsymbol{x}_0 = 0$, and there exists $\boldsymbol{u}(t)$ such that we can reach any $\boldsymbol{x}_f \in \mathbb{R}^n$; therefore by definition, $\bar{\mathcal{R}}_T(0) = \mathbb{R}^n$

Coordinate and Feedback Transformations

- Consider the coordinate transformation $\boldsymbol{z} = \boldsymbol{P}^{-1}\boldsymbol{x} \implies \dot{\boldsymbol{z}} = \boldsymbol{P}^{-1}\dot{\boldsymbol{x}}$

$$\begin{aligned} &= \boldsymbol{P}^{-1}(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}) \\ &= \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}\boldsymbol{z} + \boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{u} \end{aligned}$$

  - Therefore the coordinate transform does $(\boldsymbol{A}, \boldsymbol{B}) \to (\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}, \boldsymbol{P}^{-1}\boldsymbol{B})$
- A *feedback transformation* is $\boldsymbol{u} = \boldsymbol{K}\boldsymbol{x} + \boldsymbol{v}$ where $\boldsymbol{v}$ is the new input, for some feedback matrix $\boldsymbol{K}$ (now the system's input contains feedback based on its state)
  - $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}(\boldsymbol{K}\boldsymbol{x} + \boldsymbol{v}) = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K})\boldsymbol{x} + \boldsymbol{B}\boldsymbol{v}$
  - The feedback transform does $(\boldsymbol{A}, \boldsymbol{B}) \to (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}, \boldsymbol{B})$
- We will see that coordinate and feedback transformations do not affect the controllability of a system
  - This is useful because we can see the system under a different transformation, which may lead to more insights, and obtain information applicable to the original system

Theorem

For any nonsingular $\boldsymbol{P}$, the system $(\boldsymbol{A}, \boldsymbol{B})$ is completely controllable if and only if $(\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}, \boldsymbol{P}^{-1}\boldsymbol{B})$ is completely controllable.
For any $\boldsymbol{K}$, the system $(\boldsymbol{A}, \boldsymbol{B})$ is completely controllable if and only if $(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}, \boldsymbol{B})$ is completely controllable.

In other words, controllability is invariant under coordinate and feedback transformations.

- Proof for coordinate transform invariance:
  - From the lemma, completely controllable $(\boldsymbol{A}, \boldsymbol{B})$ means $\mathcal{R}(L_c) = \mathbb{R}^n$
  - Consider the transformed $\tilde{L}_c(\boldsymbol{u}(\cdot)) = \int_0^T e^{\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}(T-\tau)}\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau$ so $\mathcal{R}(\tilde{L}_c) = \mathbb{R}^n$ if and only if the transformed system is controllable
  - The idea is that L_c and $\tilde{L}_c$ are related by a nonsingular matrix, so they should have the same range space (similar to how $\boldsymbol{B}$ and $\boldsymbol{P}\boldsymbol{B}$ have the same range space for nonsingular $\boldsymbol{P}$)

Controllability Matrix

Theorem

Cayley-Hamilton Theorem: Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ have characteristic polynomial

$$\det(s\boldsymbol{I} - \boldsymbol{A}) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$$

then $\boldsymbol{A}$ satisfies

$$\boldsymbol{A}^n + a_{n-1}\boldsymbol{A}^{n-1} + \cdots + a_0\boldsymbol{I} = \boldsymbol{0}_{n\times n}$$

i.e. every square matrix satisfies its own characteristic equation. This allows us to express $\boldsymbol{A}^n$ and any higher powers of $\boldsymbol{A}$ as a linear combination of $\boldsymbol{I}, \boldsymbol{A}, \dots, \boldsymbol{A}^{n-1}$.

Definition

The *controllability matrix* for a system $(\boldsymbol{A}, \boldsymbol{B})$ is defined as

$$\boldsymbol{Q}_c = \begin{bmatrix}\boldsymbol{B} & \boldsymbol{AB} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B}\end{bmatrix} \in \mathbb{R}^{n\times nm}$$

The system $(\boldsymbol{A}, \boldsymbol{B})$ is completely controllable if and only if $\mathcal{R}(\boldsymbol{Q}_c) = \mathbb{R}^n$, or $\operatorname{rank}(\boldsymbol{Q}_c) = n$.

- Proof ($\operatorname{rank}(\boldsymbol{Q}_c) = n \implies (\boldsymbol{A}, \boldsymbol{B})$ is controllable):
  - Proof by contradiction: first assume the negation of the statement, i.e. let $\operatorname{rank}(\boldsymbol{Q}_c) = n$ but $(\boldsymbol{A}, \boldsymbol{B})$ not controllable; we will show that this shows $\operatorname{rank}(\boldsymbol{Q}_c) \neq n$, leading to a contradiction
  - Let $\hat{L}_c$ such that $L_c(\boldsymbol{u}(\cdot)) = \int_0^T e^{\boldsymbol{A}(T-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau = e^{\boldsymbol{A}T}\int_0^T e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau = e^{\boldsymbol{A}T}\hat{L}_c(\boldsymbol{u}(\cdot))$
    * $(\boldsymbol{A}, \boldsymbol{B})$ not controllable means $\dim(\mathcal{R}(L_c)) < n$
    * Since $e^{\boldsymbol{A}T}$ is always invertible, $\operatorname{rank}(L_c) = \operatorname{rank}(\hat{L}_c) \implies \dim(\mathcal{R}(\hat{L}_c)) < n$
  - $\mathcal{R}(\hat{L}_c)$ has an orthogonal component $\mathcal{R}(\hat{L}_c)^\perp$ where $\mathcal{R}(\hat{L}_c)^\perp \oplus \mathcal{R}(\hat{L}_c) = \mathbb{R}^n$, and $\dim(\mathcal{R}(\hat{L}_c)) < n$ means there exists a nonzero $\boldsymbol{v} \in \mathcal{R}(\hat{L}_c)^\perp$ orthogonal to all elements in $\hat{L}_c$
  - Then for any any piecewise continuous $\boldsymbol{u}(\cdot) : [0, T] \mapsto \mathbb{R}^m$, we have $\boldsymbol{v}^T\int_0^T e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau = 0$
    * Consider the control input $\boldsymbol{u}^{i,s}(t) = \begin{cases}\boldsymbol{e}_i & t \in [0, s] \\ 0 & t \in (s, T])\end{cases}$ for $s \in [0, T]$ and $i = 1, \dots, m$
      - Note $\boldsymbol{e}_i$ denotes a vector with a 1 in the ith element and all 0s everywhere else
      - This input picks out the ith column of $\boldsymbol{B}$ for $t \in [0, s]$
    * Therefore $0 = \boldsymbol{v}^T\int_0^T e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}^{i,s}(\tau)\,\mathrm{d}\tau = \boldsymbol{v}^T\int_0^s e^{-\boldsymbol{A}\tau}\boldsymbol{b}_i\mathrm{d}\tau$ holds $\forall s \in [0, T]$
      - $\frac{\mathrm{d}}{\mathrm{d}s}\left(\boldsymbol{v}^T\int_0^s e^{-\boldsymbol{A}\tau}\boldsymbol{b}_i\mathrm{d}\tau\right) = \boldsymbol{v}^Te^{-\boldsymbol{A}s}\boldsymbol{b}_i = 0$, evaluate at $s = 0$ gives us $\boldsymbol{v}^T\boldsymbol{b}_i = 0$
      - $\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\boldsymbol{v}^T\int_0^s e^{-\boldsymbol{A}\tau}\boldsymbol{b}_i\mathrm{d}\tau\right) = -\boldsymbol{v}^T\boldsymbol{A}e^{-\boldsymbol{A}s}\boldsymbol{b}_i = 0$, again at $s = 0$ gives $\boldsymbol{v}^T\boldsymbol{A}\boldsymbol{b}_i = 0$
      - Do this for up to the nth derivative, then $\boldsymbol{v}^T\boldsymbol{A}^k\boldsymbol{b}_i = 0$ for all $k = 0, \dots, n-1$
    * Altogether $\boldsymbol{v}^T\begin{bmatrix}\boldsymbol{b}_i & \boldsymbol{A}\boldsymbol{b}_i & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{b}_i\end{bmatrix} = \boldsymbol{0} \in \mathbb{R}^{1\times n}$
    * Repeat for all i, then $\boldsymbol{v}^T\begin{bmatrix}\boldsymbol{B} & \boldsymbol{AB} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B}\end{bmatrix} = \boldsymbol{0} \in \mathbb{R}^{1\times nm}$
  - Therefore we've shown there exists a nonzero $\boldsymbol{v}$ where $\boldsymbol{v}^T\boldsymbol{Q}_c = \boldsymbol{0}$, meaning the rows of $\boldsymbol{Q}_c$ are linearly dependent; since row rank equals column rank, $\operatorname{rank}(\boldsymbol{Q}_c) < n$
- Example: RLC circuit with $\boldsymbol{A} = \begin{bmatrix}0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L}\end{bmatrix}, \boldsymbol{B} = \begin{bmatrix}0 \\ \frac{1}{L}\end{bmatrix}$

- $\boldsymbol{Q}_c = \begin{bmatrix}\boldsymbol{B} & \boldsymbol{AB}\end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{LC} \\ \frac{1}{L} & -\frac{R}{L^2} \end{bmatrix}$
- For all nonzero L, C this matrix has rank 2, therefore this system is always completely controllable