Lecture 12, Oct 10, 2025

Invariant Subspaces and the Representation Theorem

Definition

A subspace $\mathcal{V} \in \mathbb{R}^n$ is $\boldsymbol{A}$-*invariant* for $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ if and only if

$$\forall \boldsymbol{x} \in \mathcal{V},\ \boldsymbol{A}\boldsymbol{x} \in \mathcal{V}$$

i.e. any vector in the subspace stays within the subspace under a linear transformation $\boldsymbol{A}$. We denote this by $\boldsymbol{A}\mathcal{V} \subseteq \mathcal{V}$.

- Note this is equivalent to $\forall \boldsymbol{x}_0 \in \mathcal{V}, e^{\boldsymbol{A}t}\boldsymbol{x}_0 \in \mathcal{V}$
- Some examples:
  - $\mathcal{N}(\boldsymbol{A}), \mathcal{R}(\boldsymbol{A})$ are both $\boldsymbol{A}$-invariant
  - If $\boldsymbol{w}_1, \dots, \boldsymbol{w}_n \in \mathbb{R}^n$ are eigenvectors of $\mathcal{N}$, then $\operatorname{span} \boldsymbol{w}_1, \dots, \boldsymbol{w}_n$ is $\boldsymbol{A}$-invariant

Theorem

Representation theorem: Let $\mathcal{X}$ be a finite dimensional vector space over $\mathbb{F}$ ($\dim(\mathcal{X}) = n$) and let $L : \mathcal{X} \mapsto \mathcal{X}$ be a linear map, and let $\mathcal{V}$ be an L-invariant subspace of $\mathcal{X}$ ($\dim(\mathcal{V}) = k$). Then there exists a basis $\{\, \boldsymbol{x}^1, \dots, \boldsymbol{x}^n \,\}$ for $\mathcal{X}$ such that the matrix representation of L in this basis has the form

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{0}_{(n-k)\times k} & \boldsymbol{A}_{22} \end{bmatrix} \qquad \boldsymbol{A}_{11} \in \mathbb{F}^{k \times k}, \boldsymbol{A}_{12} \in \mathbb{F}^{k \times (n-k)}, \boldsymbol{A}_{22} \in \mathbb{F}^{(n-k)\times(n-k)}$$

- Note that if L has a matrix representation $\boldsymbol{B}$ in the standard basis, then $\boldsymbol{A} = \boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P}$, where $\boldsymbol{P} = \begin{bmatrix} \boldsymbol{x}^1, \dots, \boldsymbol{x}^n \end{bmatrix}$
- Proof:
  - $\mathcal{V}$ is a subspace so it has an independent complement $\mathcal{W}$
  - Let $\{\, \boldsymbol{v}^1, \dots, \boldsymbol{v}^k \,\}$ be a basis for $\mathcal{V}$ and $\{\, \boldsymbol{v}^{k+1}, \dots, \boldsymbol{v}^n \,\}$ be a basis for $\mathcal{W}$, then $\{\, \boldsymbol{v}^1, \dots, \boldsymbol{v}^n \,\}$ is a basis for $\mathcal{X}$
  - $\mathcal{V}$ is L-invariant, so $L(\boldsymbol{v}^i) \in \mathcal{V}$ for $i = 1, \dots, k$ so we can express each $L(\boldsymbol{v}^i) = \sum_{i=1}^{k} a_{ji}\boldsymbol{v}^j + \sum_{l=k+1}^{n} 0\boldsymbol{v}^l$
    - For $i = k+1, \dots, n$ we no longer have $L(\boldsymbol{v}^i)$ since $\mathcal{W}$ is not L-invariant, so for these terms the second sum does not have all zeros
  - Recall column i of the matrix representation of L in this basis are the coordinates of $L(\boldsymbol{v}^i)$, so columns $i = 1, \dots, k$ have the form $\begin{bmatrix} a_{1i} \\ \vdots \\ a_{ki} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, and the rest of the columns are nonzero in general
  - Putting it all together, we get the form of $\boldsymbol{A}$ stated in the theorem
- The representation theorem allows us to split up a linear map into parts that are invariant and parts that are not
- Consider $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ and $\mathcal{V}$ as an $\boldsymbol{A}$-invariant subspace of $\mathbb{R}^n$, then the representation theorem tells us that there exists a basis $\{\, \boldsymbol{v}^1, \dots, \boldsymbol{v}^n \,\}$ of $\mathbb{R}^n$ such that the matrix representation of $\boldsymbol{A}$ has the form $\hat{\boldsymbol{A}} = \begin{bmatrix} \hat{\boldsymbol{A}}_{11} & \hat{\boldsymbol{A}}_{12} \\ \boldsymbol{0} & \hat{\boldsymbol{A}}_{22} \end{bmatrix}$
  - Now let $\boldsymbol{P} = \begin{bmatrix} \boldsymbol{v}^1 & \cdots & \boldsymbol{v}^n \end{bmatrix}$, then $\boldsymbol{A}\boldsymbol{P} = \boldsymbol{P}\hat{\boldsymbol{A}}$, i.e. $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \hat{\boldsymbol{A}}$

* $\boldsymbol{A}\boldsymbol{v}^j = \sum_{i=1}^{n} \hat{a}_{ij}\boldsymbol{v}^i = \begin{bmatrix}\boldsymbol{v}^1 & \cdots & \boldsymbol{v}^n\end{bmatrix} \begin{bmatrix}\hat{a}_{ij} \\ \vdots \\ \hat{a}_{nj}\end{bmatrix} = \boldsymbol{P} \begin{bmatrix}\hat{a}_{ij} \\ \vdots \\ \hat{a}_{nj}\end{bmatrix}$
* Recall that for a matrix representation of a linear map, column i contains the coordinates of the i-th basis vector after transformation by the linear map
  - This means $\hat{a}_{ij}$ are the coordinates of $\boldsymbol{A}\boldsymbol{v}^j$ with respect to our basis
  - The last column vector here is the jth column of $\hat{\boldsymbol{A}}$
* Repeat this for every column

– Let $\boldsymbol{z} = \boldsymbol{P}^{-1}\boldsymbol{x}$ so $\dot{\boldsymbol{z}} = \boldsymbol{P}^{-1}\dot{\boldsymbol{x}} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}\boldsymbol{z} = \hat{\boldsymbol{A}}\boldsymbol{z}$

– Then $\begin{bmatrix}\dot{\boldsymbol{z}}^1 \\ \dot{\boldsymbol{z}}^2\end{bmatrix} = \begin{bmatrix}\hat{\boldsymbol{A}}_{11} & \hat{\boldsymbol{A}}_{12} \\ \boldsymbol{0} & \hat{\boldsymbol{A}}_{22}\end{bmatrix} \begin{bmatrix}\boldsymbol{z}^1 \\ \boldsymbol{z}^2\end{bmatrix}$

– Now $\dot{\boldsymbol{z}}^1 = \hat{\boldsymbol{A}}_{11}\boldsymbol{z}^1 + \hat{\boldsymbol{A}}_{12}\boldsymbol{z}^2$ and $\dot{\boldsymbol{z}}^2 = \hat{\boldsymbol{A}}_{22}\boldsymbol{z}^2$

– Notice now that the $\boldsymbol{z}^2$ subsystem is decoupled

– We will later make use of this to define the Kalman decomposition and the notion of stabilizability