Lecture 10, Oct 1, 2025

Linear Algebra Concepts

Definition

A *vector space* $\mathcal{X}$ over a field $\mathbb{F}$ (can be e.g. $\mathbb{R}$ or $\mathbb{C}$) is a set of elements (vectors) with 2 operations: vector addition (between two elements of $\mathcal{X}$) and scalar multiplication (between an element of $\mathcal{X}$ and an element of the field $\mathbb{F}$), with the properties:

- Closure: $v + w \in \mathcal{X}$ and $\lambda v \in \mathcal{X}$
- Commutativity: $v + w = w + v$
- Associativity: $(v + w) + z = v + (w + z)$ and $(\lambda\mu)v = \lambda(\mu v)$
- Additive identity: $\exists \bar{0} \in \mathcal{X}$ such that $v + \bar{0} = v$
- Multiplicative identity: $\exists 1 \in \mathbb{F}$ and $0 \in \mathbb{F}$ such that $1v = v$ and $0v = \bar{0}$
- Additive inverse: $\exists(-v) \in \mathcal{X}$ such that $v + (-v) = \bar{0}$
- Distributivity: $\lambda(v + w) = \lambda v + \lambda w$ and $(\lambda + \mu)v = \lambda v + \mu v$

Definition

Let $\mathcal{X}$ be a vector space over $\mathbb{F}$ and let $x_1, \dots, x_m \in \mathcal{X}$, then the *span* of these vectors is

$$\operatorname{span}\{\, x_1, \dots, x_m \,\} = \left\{ \sum_{i=1}^{m} c_i x_i \;\middle|\; c_i \in \mathbb{F} \right\}$$

i.e. it is the set formed by all possible linear combinations of vectors in the set.

Definition

A vector space $\mathcal{X}$ is *finite-dimensional* if it can be expressed as

$$\mathcal{X} = \operatorname{span}\{\, x_1, \dots, x_m \,\}$$

where $m \in \mathbb{N}$ is a finite integer.

The smallest m that satisfies this relation is the *dimension* of $\mathcal{X}$.

- Some examples: $\mathbb{R}^n, \mathbb{R}^{m\times n}$, and the vector space of n-th degree polynomials P^n are all finite dimensional; but the vector space of square-integrable functions in $[a, b]$, $L^2([a, b])$ is not finite dimensional

Definition

A set of vectors $\{\, x_1, \dots, x_n \,\}$ is *linearly independent* if

$$\forall c_1, \dots, c_m \in \mathbb{F}, \sum_{i=1}^{m} c_i x_i = \bar{0} \iff c_1 = \dots = c_m = 0$$

i.e. the only linear combination of the vectors to get the zero vector is all zeros.

Definition

A set of vectors $\{\, x_1, \dots, x_m \,\} \subseteq \mathcal{X}$ is a *basis* for $\mathcal{X}$ if $\mathcal{X} = \operatorname{span}\{\, x_1, \dots, x_m \,\}$ and all vectors in the set are linearly independent.

Definition

Let $\mathcal{X}$ be a vector space over $\mathbb{F}$ and let $\{ x_1, \ldots, x_m \}$ be a basis for $\mathcal{X}$; then any vector $v \in \mathcal{X}$ can be written as $v = c_1 x_1 + \cdots + c_m x_m$, where

$$\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \in \mathbb{F}^m$$

is the *coordinate representation* of v under this basis. It can be shown that the values of $c_1, \ldots, c_m$ are uniquely determined by v and the basis $\{ x_1, \ldots, x_m \}$.

Definition

A subset $\mathcal{V} \subseteq \mathcal{X}$ is a *subspace* if it is closed, i.e. $x, y \in \mathcal{V}, \lambda \in \mathbb{F} \implies x + \lambda y \in \mathcal{V}$ and if it contains the zero vector, $\bar{0} \in \mathcal{V}$.

The *direct sum* of two subspaces is

$$\mathcal{V} \oplus \mathcal{W} = \{ v + w \mid v \in \mathcal{V}, w \in \mathcal{W} \}$$

which can be shown to be another subspace.

Definition

Two subspaces $\mathcal{V}, \mathcal{W}$ of $\mathcal{X}$ are *independent* if $\mathcal{V} \cap \mathcal{W} = \{ \bar{0}_{\mathcal{X}} \}$, i.e. their intersection contains only zero.

Theorem

Let $\mathcal{V}$ be a subspace of the vector space $\mathcal{X}$, then $\mathcal{V}$ has an *independent complement*, which is another subspace which is independent from $\mathcal{V}$ and sums with $\mathcal{V}$ to get the entirety of $\mathcal{X}$, i.e.

$$\exists \mathcal{W} \subseteq \mathcal{X} \text{ s.t. } \mathcal{V} \cap \mathcal{W} = \{ \bar{0}_{\mathcal{X}} \}, \mathcal{V} \oplus \mathcal{W} = \mathcal{X}$$

Not to be confused with an orthogonal complement.

Definition

Let $\mathcal{X}$ be a vector space over $\mathbb{R}$. An *inner product* on $\mathcal{X}$ is an operation $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R}$ with the following properties:

1. $\langle x, y \rangle = \langle y, x \rangle$ (this property requires a conjugate for $\mathbb{C}$)
2. $\langle \lambda x_1 + x_2, y \rangle = \lambda \langle x_1, y \rangle + \langle x_2, y \rangle$
3. $\langle x, x \rangle \geq 0$
4. $\langle x, x \rangle = 0 \iff x = \bar{0}$

If $\langle x, y \rangle = 0$, then x and y are *orthogonal* under the inner product $\langle \cdot, \cdot \rangle$.

Combining a vector space with an inner product, $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ is called an *inner product space*.

- For example, $\mathbb{R}^n$ and $\langle x, y \rangle = x^T y$ is an inner product space
  - On $L^2([a, b])$ we can define an inner product $\langle f, g \rangle = \int_a^b f(\tau) g(\tau) \, \mathrm{d}\tau$

Definition

Let $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ be an inner product space and let $\mathcal{V} \subseteq \mathcal{X}$ be a subspace. Then the *orthogonal complement* of $\mathcal{V}$ in $\mathcal{X}$ is the subspace

$$\mathcal{V}^{\perp} = \{\, w \in \mathcal{X} \mid \langle w, v \rangle = 0,\, \forall v \in \mathcal{V} \,\}$$

Note $\mathcal{V} \oplus \mathcal{V}^{\perp} = \mathcal{X}$.