Lecture 1, Sep 3, 2025

Theory of Linear Control Systems

- *Dynamical systems* (aka *systems* or *dynamical processes*) are mathematical models that describe how quantities of interest evolve over time
  - At a high level: input $u(t)$, output $y(t)$
- We are interested in the theory of *analysis* (how is the system behaving?) and *control* (how can we make the system behave well/better?) for *linear time-invariant* (LTI) systems in continuous time

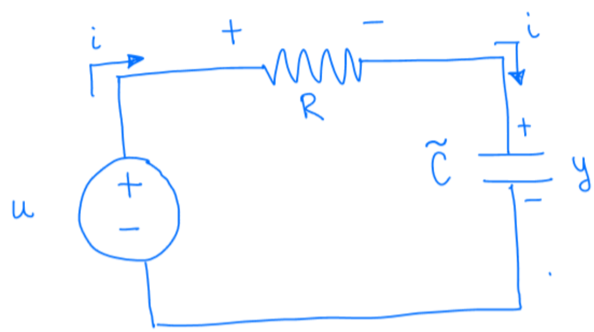

Figure 1: Circuit example of a dynamical system.

- Consider the circuit above; let y be the voltage across the capacitor and u be the input voltage
  - $i = \tilde{C}\frac{\mathrm{d}y}{\mathrm{d}t}$
  - By KVL: $u - iR - y = 0 \implies u - R\tilde{C}\frac{\mathrm{d}y}{\mathrm{d}t} - y = 0$
  - Rearrange: $\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{1}{R\tilde{C}}y + \frac{1}{R\tilde{C}}u$
  - We now have an ODE representing the LTI system

Definition

A *linear time-invariant* system in *state-space* form is represented by the following:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), t \geq 0$$
$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)$$

where

$$\boldsymbol{x} : [0, \infty) \in \mathbb{R}^n (\text{state})$$
$$\boldsymbol{u} : [0, \infty) \in \mathbb{R}^m (\text{input})$$
$$\boldsymbol{y} : [0, \infty) \in \mathbb{R}^p (\text{output})$$

The first is known as the *state equation* while the second is the *measurement equation.*

- $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is the *system* matrix.
- $\boldsymbol{B} \in \mathbb{R}^{n \times m}$ is the *input* matrix.
- $\boldsymbol{C} \in \mathbb{R}^{p \times n}$ is the *output* matrix.
- $\boldsymbol{D} \in \mathbb{R}^{p \times m}$ is the *feedforward* matrix.

- For our example system: $x = y, \dot{x} = -\frac{1}{R\tilde{C}}x + \frac{1}{R\tilde{C}}u, y = x + u$ (note in this example we can fully measure the state)
  - $A = -\frac{1}{R\tilde{C}}$
  - $B = \frac{1}{R\tilde{C}}$

  - $C = 1$
  - $D = 0$
- Within dynamical systems we have several forms of representation:
  - Model-based representation (for systems represented as ODEs)
    * We are interested in continuous time, deterministic (no noise), LTI systems
  - Data-based representation (for streams of data)
  - Computer-based representation (for complicated systems that we cannot write down)