1 Dynamical Systems

LTI System: $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)$ where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ (system), $\boldsymbol{B} \in \mathbb{R}^{n\times m}$ (input), $\boldsymbol{C} \in \mathbb{R}^{p\times n}$ (output) $\boldsymbol{D} \in \mathbb{R}^{p\times m}$ (feedforward)

TF to SS: $G(s) = N(s)/D(s) \implies D(s)V(s) = U(s), N(s)V(s) = Y(s)$; inverse Laplace & let x be v and its derivatives, rearrange to get state eqn. from first and measurement eqn. from second.

SS to TF: $\boldsymbol{G}(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D} \in \mathbb{R}^{p\times m}$

Linearization: Let $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}), \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u})$, let $\delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}^*, \delta\boldsymbol{u} = \boldsymbol{u} - \boldsymbol{u}^*$, the linearized system about $(\boldsymbol{x}^*, \boldsymbol{u}^*)$ is ($\boldsymbol{f}(\boldsymbol{x}^*, \boldsymbol{u}^*) = 0$ for eqm. condition):

$$\delta\dot{\boldsymbol{x}} = \boldsymbol{A}\delta\boldsymbol{x} + \boldsymbol{B}\delta\boldsymbol{u} + \boldsymbol{f}(\boldsymbol{x}^*, \boldsymbol{u}^*) \qquad \boldsymbol{A} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}^*, \boldsymbol{u}^*) \qquad \boldsymbol{B} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{x}^*, \boldsymbol{u}^*)$$

$$\boldsymbol{y} = \boldsymbol{C}\delta\boldsymbol{x} + \boldsymbol{D}\delta\boldsymbol{u} \qquad \boldsymbol{C} = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}(\boldsymbol{x}^*, \boldsymbol{u}^*) \qquad \boldsymbol{D} = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{u}}(\boldsymbol{x}^*, \boldsymbol{u}^*)$$

Solutions: $\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0 + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau$

For autonomous ($\boldsymbol{u} = 0$) systems, the matrix exponential is the unique soln.

2 Matrix Exponential & Diagonalization

Matrix Exponential: $e^{\boldsymbol{A}} = \sum_{k=0}^{\infty} \frac{1}{k!}\boldsymbol{A}^k$ has the following properties:

1. For all invertible $\boldsymbol{P} \in \mathbb{R}^{n\times n}$, $e^{\boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^{-1}} = \boldsymbol{P}e^{\boldsymbol{A}}\boldsymbol{P}^{-1}$.
2. $e^{\boldsymbol{A}+\boldsymbol{B}} = e^{\boldsymbol{A}}e^{\boldsymbol{B}} = e^{\boldsymbol{B}}e^{\boldsymbol{A}}$ if $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{A}$.
3. Matrix exponential is always invertible: $(e^{\boldsymbol{A}})^{-1} = e^{-\boldsymbol{A}}$.
4. $\forall t \in \mathbb{R}, \frac{\mathrm{d}}{\mathrm{d}t}e^{\boldsymbol{A}t} = \boldsymbol{A}e^{\boldsymbol{A}t} = e^{\boldsymbol{A}t}\boldsymbol{A}$.

From Laplace Transform: $\boldsymbol{e^{At}} = \mathcal{L}^{-1}\left\{(s\boldsymbol{I} - \boldsymbol{A})^{-1}\right\}$

Diagonalization: $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{\Lambda}$ where $\boldsymbol{P}$ has e'vects, $\boldsymbol{\Lambda}$ is diagonal with e'vals. Possible if and only if $\boldsymbol{A}$ has n lin. indep. e'vects, then $e^{\boldsymbol{A}} = \boldsymbol{P}e^{\boldsymbol{\Lambda}}\boldsymbol{P}^{-1}$.

If $\boldsymbol{A}$ has n distinct eigenvalues or if $\boldsymbol{A}$ is symmetric, it's always diagonalizable.

Complex Eigenvalues: Conjugate pairs; can find real $\tilde{\boldsymbol{P}}\tilde{\boldsymbol{\Lambda}}\tilde{\boldsymbol{P}}^{-1} = \boldsymbol{A}$:

$$\tilde{\boldsymbol{P}} = \begin{bmatrix}\boldsymbol{v}_r & \mathrm{Re}(\boldsymbol{v}_c) & \mathrm{Im}(\boldsymbol{v}_c)\end{bmatrix} \qquad \tilde{\boldsymbol{\Lambda}} = \begin{bmatrix}\lambda_r & 0 & 0 \\ 0 & \mathrm{Re}(\lambda_c) & \mathrm{Im}(\lambda_c) \\ 0 & -\mathrm{Im}(\lambda_c) & \mathrm{Re}(\lambda_c)\end{bmatrix}$$

$$\exp\left(\begin{bmatrix}a & b \\ -b & a\end{bmatrix}t\right) = e^{at}\begin{bmatrix}\cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt)\end{bmatrix}$$

For real $\boldsymbol{v}_r, \lambda_r$, complex $\boldsymbol{v}_c, \lambda_c$. Rotation CW if $b > 0$, CCW if $b < 0$.

Algebraic Multiplicity: m_i is the number of times λ_i appears as an eigenvalue of $\boldsymbol{A}$, i.e. the power of $s - \lambda_i$ in $\det(s\boldsymbol{I} - \boldsymbol{A})$. $\sum_i m_i = n$.

Geometric Multiplicity: $l_i = \dim(\mathcal{N}(\lambda_i\boldsymbol{I} - \boldsymbol{A}))$, i.e. the number of indep. e'vects corresponding to λ_i. $l_i \le m_i$; $l_i = m_i$ if and only if $\boldsymbol{A}$ is diagonalizable.

Generalized Eigenvector: $(\lambda\boldsymbol{I} - \boldsymbol{A})^k\boldsymbol{v} = \boldsymbol{0}, k \in \mathbb{N}$. For each (true) e'vect $\boldsymbol{e} = \boldsymbol{v}_1$ for a given λ, we can form a chain $(\lambda\boldsymbol{I} - \boldsymbol{A})\boldsymbol{v}_{k+1} = -\boldsymbol{v}_k$ until we have m_i gen. e'vects in total, and form $\boldsymbol{P}$ by grouping gen. e'vect chains corresponding to each true e'vect, then grouping e'vects for each e'val.

Jordan Normal Form: For nondiagonalizable $\boldsymbol{A}$, $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{J}$ where

$$\boldsymbol{J} = \begin{bmatrix}\boldsymbol{J}_{\lambda_1} & & \\ & \ddots & \\ & & \boldsymbol{J}_{\lambda_k}\end{bmatrix} \qquad \boldsymbol{J}_{\lambda_i} = \begin{bmatrix}\boldsymbol{J}_{\lambda_i}^1 & & \\ & \ddots & \\ & & \boldsymbol{J}_{\lambda_i}^{l_i}\end{bmatrix} \qquad \boldsymbol{J}_{\lambda_i}^j = \begin{bmatrix}\lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i\end{bmatrix}$$

Each $\boldsymbol{J}_{\lambda_i} \in \mathbb{C}^{m_i\times m_i}$ corresponds to a distinct e'val. Each $\boldsymbol{J}_{\lambda_i}^j$ corresponds to the chain of generalized e'vects for one true e'vect of that eigenspace.

$$e^{\boldsymbol{J}_{\lambda_i}t} = \begin{bmatrix}e^{\boldsymbol{J}_{\lambda_i}^1} & & \\ & \ddots & \\ & & e^{\boldsymbol{J}_{\lambda_i}^{l_i}}\end{bmatrix} \qquad e^{\boldsymbol{J}_{\lambda_i}^j t} = e^{\lambda_i t}\begin{bmatrix}1 & t & \cdots & \frac{t^{p-1}}{(p-1)!} \\ & \ddots & \ddots & \vdots \\ & & 1 & t \\ & & & 1\end{bmatrix}$$

3 System Behaviour & Stability

Modal Decomposition: Let $\boldsymbol{z}(t) = \boldsymbol{P}^{-1}\boldsymbol{x}(t)$, then $\dot{\boldsymbol{z}} = \boldsymbol{\Lambda}\boldsymbol{z}$, and each $z_i(t) = e^{\lambda_i t}z_i(0)$. Decompose the solution into modes:

$$\boldsymbol{x}(t) = \boldsymbol{P}\boldsymbol{z}(t) = \sum_{i=1}^{n}\boldsymbol{v}_i\boldsymbol{z}_i(t) = \sum_{i=1}^{n}\boldsymbol{v}_i e^{\lambda_i t}z_i(0) = \sum_{i=1}^{n}\boldsymbol{h}_i(t)$$

Geometrically, any initial condition is broken into a linear combination of the modes; in z-space, the modes become axis-aligned.

Stability: $\forall\boldsymbol{x}_0 \in \mathbb{R}^n, \exists M < \infty$ s.t. $\|\boldsymbol{x}(t)\| \le M, \forall t \ge 0$ (solns. bounded)

Asymptotic Stability: $\forall\boldsymbol{x}_0 \in \mathbb{R}^n, \lim_{t\to\infty} x(t) = \boldsymbol{0}$. Implies BIBO and input-output stability.

BIBO Stability: For $\boldsymbol{x}_0 = \boldsymbol{0}$, $\boldsymbol{u}(t)$ bounded $\implies \boldsymbol{y}(t)$ also bounded. If $G(s)$ is stable, the system is BIBO stable but not always I/O stable.

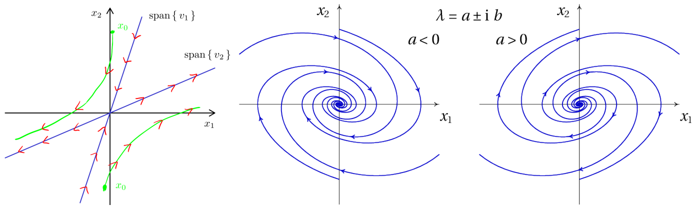
Input-Output Stability: For any $\boldsymbol{x}_0 \in \mathbb{R}^n$, $\boldsymbol{u}(t)$ bounded $\implies \boldsymbol{y}(t)$ also bounded. Implies BIBO stability.

Lyapunov Equation: If there exists positive definite $\boldsymbol{P}$ such that $\boldsymbol{Q} = -\boldsymbol{A}^T\boldsymbol{P} - \boldsymbol{P}\boldsymbol{A}$ is positive definite, then $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ is asymptotically stable.

Relation to Eigenvalues: The system is asymptotically stable if and only if $\mathrm{Re}(\lambda_i) < 0$. If any $\mathrm{Re}(\lambda_i) > 0$, it is unstable. For e'vals where $\mathrm{Re}(\lambda_i) = 0$, the system is stable if and only if $m_i = l_i$.

System Behaviour According to Eigenvalues:

1. Real nonzero eigenvalues: Stable node if $\lambda_i < 0$, unstable node if $\lambda_i > 0$, saddle if mixed signs.
2. Complex eigenvalues: Stable focus if $\mathrm{Re}(\lambda_i) < 0$, unstable focus if $\mathrm{Re}(\lambda_i) > 0$, centre if $\mathrm{Re}(\lambda_i) = 0$.
3. Zero eigenvalues: Solutions converge or diverge from an equilibrium line.



4 Linear Transformations

Vector Space: Set of vectors, field $\mathbb{F}$, operations addition and multiplication, satisfying: 1. Closure, 2. Addition Commutativity, 3. Associativity, 4. Identity Elements, 5. Additive Inverse, and 6. Distributivity.

Span: $\mathrm{span}\{x_1, \ldots, x_m\} = \{\sum_{i=1}^m c_i x_i \mid c_i \in \mathbb{F}\}$

Dimension: Smallest m such that $\mathcal{X} = \mathrm{span}\{x_1, \ldots, x_m\}$.

Linear Independence: $\forall c_1, \ldots, c_m \in \mathbb{F}, \sum_{i=1}^m c_i x_i = \theta \iff c_i = 0, \forall i$

Basis: $\mathcal{B} = \{x_1, \ldots, x_m\} \subseteq \mathcal{X}$ is a basis if $\mathcal{X} = \mathrm{span}\,\mathcal{B}$ and $\mathcal{B}$ is lin. indep.

Coordinate Representation: $v = \sum_{i=1}^m c_i x_i$; unique for a given basis.

Subspace: $\mathcal{V} \subseteq \mathcal{X}$ is a subspace if it is closed and contains the zero vector.

Direct Sum: $\mathcal{V} \oplus \mathcal{W} = \{v + w \mid v \in \mathcal{V}, w \in \mathcal{W}\}$; is another subspace.

Indep. Comp.: Subspace where $\mathcal{W} \cap \mathcal{V} = \{0\}$ and $\mathcal{W} \oplus \mathcal{V} = \mathcal{X}$.

Orthogonal Comp.: $\mathcal{V}^\perp = \{w \in \mathcal{X} \mid \langle w, v\rangle = 0, \forall v \in \mathcal{V}\}$. $\mathcal{V}^\perp \oplus \mathcal{V} = \mathcal{X}$.

Injective/One-to-One: $\forall x_1, x_2 \in \mathcal{X}, f(x_1) = f(x_2) \implies x_1 = x_2$

Surjective/Onto: $\forall y \in \mathcal{Y}, \exists x \in \mathcal{X}$ s.t. $f(x) = y$

Bijective: Both injective and surjective.

Linear Map: $\forall x_1, x_2 \in \mathcal{X}, \lambda \in \mathbb{F}, L(x + \lambda y) = L(x) + \lambda L(y)$

Null Space/Kernel: $\mathcal{N}(L) = \{\boldsymbol{x} \in \mathcal{X} \mid L(\boldsymbol{x}) = \theta\}$

Range Space/Image: $\mathcal{R}(L) = \{\boldsymbol{y} \in \mathcal{Y} \mid \exists x \in \mathcal{X}, \boldsymbol{y} = L(\boldsymbol{x})\}$

Rank-Nullity: $L : \mathcal{X} \mapsto \mathcal{Y}$ satisfies $\dim(\mathcal{R}(L)) + \dim(\mathcal{N}(L)) = \dim(\mathcal{X})$.

Matrix Representation: For $L : \mathcal{X} \mapsto \mathcal{Y}$, the ith column of $\boldsymbol{A}$ contains the coordinates of $L(x^i)$ where x^i is the ith basis vector of $\mathcal{X}$, under the $\mathcal{Y}$ basis.

L is injective if and only if $\dim(\mathcal{N}(\boldsymbol{A})) = 0$. L is surjective if and only if $\mathrm{rank}(\boldsymbol{A}) = \dim(\mathcal{Y})$. L is bijective if and only if $\boldsymbol{A}$ is square and invertible.

Similarity Transform: $L : \mathbb{R}^n \mapsto \mathbb{R}^n$ defined as $L(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ expressed in basis $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ for $\mathbb{R}^n$ is $\hat{\boldsymbol{A}} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ where $\boldsymbol{P} = \begin{bmatrix}\boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n\end{bmatrix}$.

Invariant Subspace: $\mathcal{V}$ is $\boldsymbol{A}$-invariant ($\boldsymbol{A}\mathcal{V} \subseteq \mathcal{V}$) if $\forall\boldsymbol{x} \in \mathcal{V}$, $\boldsymbol{A}\boldsymbol{x} \in \mathcal{V}$.

Representation Theorem: Let $\boldsymbol{A} \in \mathbb{F}^{n\times n}$ and $\mathcal{V}$ be $\boldsymbol{A}$-invariant, $\dim\mathcal{V} = k$, then there exists a basis/coordinate transform $\boldsymbol{z} = \boldsymbol{P}^{-1}\boldsymbol{x}, \boldsymbol{P} = \begin{bmatrix}\boldsymbol{x}^1 & \cdots & \boldsymbol{x}^n\end{bmatrix}$ where $\{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k\}$ is a basis for $\mathcal{V}$, such that $\hat{\boldsymbol{A}} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{bmatrix}\hat{\boldsymbol{A}}_{11} & \hat{\boldsymbol{A}}_{12} \\ 0 & \hat{\boldsymbol{A}}_{22}\end{bmatrix}$ where $\hat{\boldsymbol{A}}_{11} \in \mathbb{F}^{k\times k}, \hat{\boldsymbol{A}}_{12} \in \mathbb{F}^{k\times(n-k)}, \hat{\boldsymbol{A}}_{22} \in \mathbb{F}^{(n-k)\times(n-k)}$.

5 Controllability and State Feedback Control

Controllability: $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$ is controllable if $\forall\boldsymbol{x}_0, \boldsymbol{x}_f \in \mathbb{R}^n, \exists$ piecewise continuous $\boldsymbol{u}(t)$ which takes the system from $\boldsymbol{x}_0 \to \boldsymbol{x}_f$ in time T:

$$\boldsymbol{x}_f = \boldsymbol{x}(T) = e^{\boldsymbol{A}T}\boldsymbol{x}_0 + \int_0^T e^{\boldsymbol{A}(T-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau = e^{\boldsymbol{A}T}\boldsymbol{x}_0 + L_c(\boldsymbol{u}(\cdot))$$

The system is controllable if and only if the *controllable subspace* $\mathcal{R}(L_c) = \mathbb{R}^n$.

Coordinate and Feedback Invariance: $(\boldsymbol{A}, \boldsymbol{B}), (\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}, \boldsymbol{P}^{-1}\boldsymbol{B})$, and $(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}, \boldsymbol{B})$ have the same controllability for all $\boldsymbol{K}$ and nonsingular $\boldsymbol{P}$.

Controllability Matrix: For $(\boldsymbol{A}, \boldsymbol{B}), \boldsymbol{Q}_c = \begin{bmatrix}\boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B}\end{bmatrix} \in \mathbb{R}^{n\times nm}$ with $\mathcal{R}(\boldsymbol{Q}_c) = \mathcal{R}(L_c)$; the system is controllable if and only if $\mathrm{rank}(\boldsymbol{Q}_c) = n$.

PBH Test (Controllability): $(\boldsymbol{A}, \boldsymbol{B})$ is completely controllable if and only if $\mathrm{rank}(\begin{bmatrix}s\boldsymbol{I} - \boldsymbol{A} & \boldsymbol{B}\end{bmatrix}) = n, \forall s \in \mathbb{C}$, or equivalently for all eigenvalues of $\boldsymbol{A}$.

Kalman Decomposition (Controllability): For $(\boldsymbol{A}, \boldsymbol{B})$ with $\mathrm{rank}(\boldsymbol{Q}_c) = k < n$, let $\boldsymbol{z} = \boldsymbol{P}^{-1}\boldsymbol{x}$, where the first k columns of $\boldsymbol{P}$ form a basis for $\boldsymbol{Q}_c$, then

$$\dot{\boldsymbol{z}} = \begin{bmatrix}\dot{\boldsymbol{z}}^1 \\ \dot{\boldsymbol{z}}^2\end{bmatrix} = \begin{bmatrix}\hat{\boldsymbol{A}}_{11} & \hat{\boldsymbol{A}}_{12} \\ \boldsymbol{0} & \hat{\boldsymbol{A}}_{12}\end{bmatrix}\begin{bmatrix}\boldsymbol{z}^1 \\ \boldsymbol{z}^2\end{bmatrix} + \begin{bmatrix}\hat{\boldsymbol{B}}_1 \\ \boldsymbol{0}\end{bmatrix}\boldsymbol{u} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}\boldsymbol{z} + \boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{u} = \hat{\boldsymbol{A}}\boldsymbol{z} + \hat{\boldsymbol{B}}\boldsymbol{u}$$

where $(\hat{\boldsymbol{A}}_{11}, \hat{\boldsymbol{B}}_1)$ forms the k-dimensional *controllable subsystem*, containing *controllable eigenvalues* that can be changed; $\hat{\boldsymbol{A}}_{22}$ contains *uncontrollable eigenvalues* which cannot be changed. The system is stabilizable if and only if all uncontrollable eigenvalues have negative real part.

Controllable Canonical Form: A single-input system $(\boldsymbol{A}, \boldsymbol{b})$ is controllable if and only if there is a coordinate transform $\boldsymbol{P} = \boldsymbol{Q}_c\boldsymbol{T}, \boldsymbol{z} = \boldsymbol{P}^{-1}\boldsymbol{x}$ where the resulting system $\dot{\boldsymbol{z}} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}\boldsymbol{z} + \boldsymbol{P}^{-1}\boldsymbol{b}\boldsymbol{u} = \tilde{\boldsymbol{A}}\boldsymbol{z} + \tilde{\boldsymbol{b}}\boldsymbol{u}$ has the form

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \qquad \tilde{\boldsymbol{b}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad \boldsymbol{T} = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

where a_i are coefficients in $\det(s\boldsymbol{I} - \boldsymbol{A})$.

Pole Assignment Problem: Find $\boldsymbol{K}$ such that $\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}$ has the desired eigenvalues (equivalent to controller $\boldsymbol{u} = \boldsymbol{K}\boldsymbol{x}$). Solvable if and only if $(\boldsymbol{A}, \boldsymbol{B})$ is controllable.

Stabilizability: $(\boldsymbol{A}, \boldsymbol{B})$ is stabilizable if there exists $\boldsymbol{K}$ such that all eigenvalues of $\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}$ have negative real part. Implied by controllability.

PBH Test (Stabilizability): $(\boldsymbol{A}, \boldsymbol{B})$ is stabilizable if and only if $\operatorname{rank}(\begin{bmatrix}\lambda\boldsymbol{I} - \boldsymbol{A} & \boldsymbol{B}\end{bmatrix}) = n$ for all eigenvalues λ of $\boldsymbol{A}$ where $\operatorname{Re}(\lambda) \geq 0$. λ is controllable if and only if $\operatorname{rank}(\begin{bmatrix}\lambda\boldsymbol{I} - \boldsymbol{A} & \boldsymbol{B}\end{bmatrix}) = n$, uncontrollable if and only if $< n$.

6 Observability and State Estimation

State Estimation Problem: Given $\boldsymbol{y}(t), \boldsymbol{u}(t), 0 \leq t \leq T$, estimate $\boldsymbol{x}(t)$ on the same interval. Equivalently, given $\boldsymbol{C}e^{\boldsymbol{A}t}\boldsymbol{x}_0, 0 \leq t \leq T$, estimate $\boldsymbol{x}_0$.

Observability: $(\boldsymbol{C}, \boldsymbol{A})$ is observable if and only if $L_o(\boldsymbol{x}_0) = \boldsymbol{C}e^{\boldsymbol{A}t}\boldsymbol{x}_0$ is injective, or equivalently the *unobservable subspace* $\mathcal{N}(L_o) = \{\bar{0}\}$. Note $L_o : \mathbb{R}^n \mapsto \mathcal{C}([0,\infty], \mathbb{R}^p)$ maps from initial conditions to functions in output space.

Observability Matrix: For a system $(\boldsymbol{C}, \boldsymbol{A})$, $\boldsymbol{Q}_o = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{A} \\ \vdots \\ \boldsymbol{C}\boldsymbol{A}^{n-1} \end{bmatrix}$ satisfies $\mathcal{N}(\boldsymbol{Q}_o) = \mathcal{N}(L_o)$, so the system is observable if and only if $\operatorname{rank}(\boldsymbol{Q}_o) = n$. This is equivalent to $\boldsymbol{Q}_c^T$ for the dual system $(\boldsymbol{A}^T, \boldsymbol{C}^T)$.

Kalman Decomposition (Observability): For unobservable $(\boldsymbol{C}, \boldsymbol{A})$ with $k = n - \operatorname{rank}(\boldsymbol{Q}_o) > 0$, let $\boldsymbol{z} = \boldsymbol{P}^{-1}\boldsymbol{x}$, where the first k columns of $\boldsymbol{P}$ form a basis for $\mathcal{N}(Q_o)$, then

$$\dot{\boldsymbol{z}} = \begin{bmatrix} \dot{\boldsymbol{z}}^1 \\ \dot{\boldsymbol{z}}^2 \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{A}}_{11} & \hat{\boldsymbol{A}}_{12} \\ 0 & \hat{\boldsymbol{A}}_{22} \end{bmatrix} \boldsymbol{z} + \begin{bmatrix} \hat{\boldsymbol{B}}_1 \\ \hat{\boldsymbol{B}}_2 \end{bmatrix} \boldsymbol{u} \qquad \boldsymbol{y} = \begin{bmatrix} 0 & \hat{\boldsymbol{C}}_2 \end{bmatrix} \boldsymbol{z} + \boldsymbol{D}\boldsymbol{u} = \boldsymbol{C}\boldsymbol{P}\boldsymbol{z} + \boldsymbol{D}\boldsymbol{u}$$

where $(\hat{\boldsymbol{C}}_2, \hat{\boldsymbol{A}}_{22})$ is the k-dimensional *observable subsystem*, containing *observable eigenvalues* that can be changed; $\hat{\boldsymbol{A}}_{11}$ contains *unobservable eigenvalues* which cannot be changed. The system is detectable if and only if all unobservable eigenvalues have negative real part.

Kalman Decomposition (General): Let $\mathcal{V}_{c\bar{o}} = \mathcal{R}(\boldsymbol{Q}_c) \cap \mathcal{N}(\boldsymbol{Q}_o)$ (controllable, unobservable), $\mathcal{V}_{co}$ s.t. $\mathcal{V}_{co} \oplus \mathcal{V}_{c\bar{o}} = \mathcal{R}(\boldsymbol{Q}_c)$ (controllable, observable), $\mathcal{V}_{\bar{c}\bar{o}}$ s.t. $\mathcal{V}_{\bar{c}\bar{o}} \oplus \mathcal{V}_{c\bar{o}} = \mathcal{N}(\boldsymbol{Q}_o)$ (uncontrollable, unobservable), and $\mathcal{V}_{\bar{c}o}$ such that all subspaces sum to $\mathbb{R}^n$ (uncontrollable, observable). Let $\boldsymbol{z} = \boldsymbol{P}^{-1}\boldsymbol{x}$, where $\boldsymbol{P}$ contains a basis for each subspace in order, then

$$\dot{\boldsymbol{z}} = \begin{bmatrix} \hat{\boldsymbol{A}}_{c\bar{o}} & * & * & * \\ 0 & \hat{\boldsymbol{A}}_{co} & 0 & * \\ 0 & 0 & \hat{\boldsymbol{A}}_{\bar{c}\bar{o}} & * \\ 0 & 0 & 0 & \hat{\boldsymbol{A}}_{\bar{c}o} \end{bmatrix} \begin{bmatrix} \boldsymbol{z}^1 \\ \boldsymbol{z}^2 \\ \boldsymbol{z}^3 \\ \boldsymbol{z}^4 \end{bmatrix} + \begin{bmatrix} \hat{\boldsymbol{B}}_{c\bar{o}} \\ \hat{\boldsymbol{B}}_{co} \\ 0 \\ 0 \end{bmatrix} \boldsymbol{u} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}\boldsymbol{z} + \boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{u}$$

$$\boldsymbol{y} = \begin{bmatrix} 0 & \hat{\boldsymbol{C}}_{co} & 0 & \hat{\boldsymbol{C}}_{\bar{c}o} \end{bmatrix} \boldsymbol{z} + \boldsymbol{D}\boldsymbol{u} = \boldsymbol{C}\boldsymbol{P}\boldsymbol{z} + \boldsymbol{D}\boldsymbol{u}$$

The controllable subsystem is $(\boldsymbol{z}^1, \boldsymbol{z}^2)$, the observable subsystem is $(\boldsymbol{z}^2, \boldsymbol{z}^4)$, and the controllable and observable subsystem is $\boldsymbol{z}^2$.

Minimal Realization: $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ has the same transfer function as its minimal realization $(\hat{\boldsymbol{A}}_{co}, \hat{\boldsymbol{B}}_{co}, \hat{\boldsymbol{C}}_{co}, \boldsymbol{D})$, which is the smallest (lowest number of states) system that can have this transfer function. Therefore if a transfer function has pole-zero cancellations, then its state-space representation is uncontrollable and/or unobservable.

State Observer: An LTI system $\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{L}(\boldsymbol{y} - \hat{\boldsymbol{y}}), \hat{\boldsymbol{y}} = \boldsymbol{C}\hat{\boldsymbol{x}} + \boldsymbol{D}\boldsymbol{u}$ which predicts a state estimate $\hat{\boldsymbol{x}}$ with error dynamics $\dot{\boldsymbol{e}} = (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\boldsymbol{e}, \boldsymbol{e} = \boldsymbol{x} - \hat{\boldsymbol{x}}$.

Detectability: $(\boldsymbol{C}, \boldsymbol{A})$ is detectable if there exists $\boldsymbol{L}$ such that all eigenvalues of $\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}$ have negative real part. Implied by observability.

Duality: For a system $(\boldsymbol{C}, \boldsymbol{A})$, define its dual system $(\boldsymbol{A}^T, \boldsymbol{C}^T)$, then the system is observable if and only if its dual system is controllable, and the system is detectable if and only if the its dual system is stabilizable.

PBH Test (Detectability): $(\boldsymbol{C}, \boldsymbol{A})$ is detectable if and only if $\operatorname{rank}\left(\begin{bmatrix} \lambda\boldsymbol{I} - \boldsymbol{A} \\ \boldsymbol{C} \end{bmatrix}\right) = n$ for all eigenvalues λ of $\boldsymbol{A}$ where $\operatorname{Re}(\lambda) \geq 0$. λ is observable if and only if the matrix has rank n, unobservable if and only if $< n$.

7 Stabilizer/Observer Design

Separation Principle: To stabilize $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ through output feedback, we can separately design a state feedback stabilizer and observer, then $\boldsymbol{u} = \boldsymbol{K}\hat{\boldsymbol{x}}$ asymptotically stabilizes the system. The closed-loop system has dynamics $\dot{\boldsymbol{x}} = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K})\boldsymbol{x} - \boldsymbol{B}\boldsymbol{K}\boldsymbol{e}, \dot{\boldsymbol{e}} = (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\boldsymbol{e}$ where $\boldsymbol{e} = \boldsymbol{x} - \hat{\boldsymbol{x}}$.

Stabilizing a Constant State: To make $\boldsymbol{x} \to \bar{\boldsymbol{x}}$, find $\bar{\boldsymbol{u}}$ s.t. $\boldsymbol{A}\bar{\boldsymbol{x}} + \boldsymbol{B}\bar{\boldsymbol{u}} = 0$, then use $\boldsymbol{u} = \boldsymbol{K}(\boldsymbol{x} - \bar{\boldsymbol{x}}) + \bar{\boldsymbol{u}} \iff \tilde{\boldsymbol{u}} = \boldsymbol{K}\tilde{\boldsymbol{x}}$ where $\boldsymbol{K}$ stabilizes the original system. Let $\tilde{\boldsymbol{x}} = \boldsymbol{x} - \bar{\boldsymbol{x}}, \tilde{\boldsymbol{u}} = \boldsymbol{u} - \bar{\boldsymbol{u}}$ then the system dynamics are $\dot{\tilde{\boldsymbol{x}}} = \boldsymbol{A}\tilde{\boldsymbol{x}} + \boldsymbol{B}\tilde{\boldsymbol{u}}$. For output feedback let $\tilde{\boldsymbol{y}} = \boldsymbol{y} - \boldsymbol{C}\bar{\boldsymbol{x}} - \boldsymbol{D}\bar{\boldsymbol{u}} = \boldsymbol{C}\tilde{\boldsymbol{x}} + \boldsymbol{D}\tilde{\boldsymbol{u}}$, then use $\tilde{\boldsymbol{y}}$ in observer to estimate $\tilde{\boldsymbol{x}}$.

7.1 State Feedback Stabilizer Design

Uncontrollable Systems: Find Kalman decomposition with coordinate transform $\boldsymbol{P}$ and verify stabilizability. Then find $\hat{\boldsymbol{K}}$ to stabilize the controllable subsystem $\hat{\boldsymbol{A}}_{11}$. Transform back to get the final gain $\boldsymbol{K} = \begin{bmatrix} \hat{\boldsymbol{K}} & 0_{m\times(n-k)} \end{bmatrix} \boldsymbol{P}^{-1}$.

Controllable Single-Input Systems: Expand $\det(s\boldsymbol{I} - \boldsymbol{A})$ to get coefficients $a_0, \ldots, a_{n-1}$. Expand the desired char. poly. for $\boldsymbol{A} + \boldsymbol{K}\boldsymbol{b}$, $(s - \lambda_1)\cdots(s - \lambda_n)$, to get coefficients $\alpha_0, \ldots, \alpha_{n-1}$. $\tilde{\boldsymbol{K}} = \begin{bmatrix} a_0 - \alpha_0 & \cdots & a_{n-1} - \alpha_{n-1} \end{bmatrix}$ is the gain for the controllable canonical form. Determine $\boldsymbol{P} = \boldsymbol{Q}_c\boldsymbol{T}$, then transform back to get the final gain $\boldsymbol{K} = \tilde{\boldsymbol{K}}\boldsymbol{P}^{-1}$.

General Controllable Systems: Determine $\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}$ in terms of the entries of $\boldsymbol{K}$ and expand $\det(s\boldsymbol{I} - (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}))$ to get coefficients. Match the coefficients against the desired coefficients $\alpha_0, \ldots, \alpha_{n-1}$ and solve for entries of $\boldsymbol{K}$.

7.2 Observer Design

Unobservable Systems: Find Kalman decomposition with coordinate transform $\boldsymbol{P}$ and verify detectability. Then find $\hat{\boldsymbol{L}}$ for the observable subsystem $\hat{\boldsymbol{A}}_{22}$. Transform back to get the final gain $\boldsymbol{L} = \boldsymbol{P}\begin{bmatrix} 0_{k\times p} \\ \hat{\boldsymbol{L}} \end{bmatrix}$.

Observable Single-Output Systems: Use the method for controllable single-input systems to find gain $\boldsymbol{K}$ for the dual system $\boldsymbol{A}^T, \boldsymbol{C}^T$, i.e. place the poles of $(\boldsymbol{A}^T + \boldsymbol{C}^T\boldsymbol{K})$, then take $\boldsymbol{L} = -\boldsymbol{K}^T$.

General Observable Systems: Same as general controllable systems.

8 Linear Quadratic Optimal Control

Cost Functional: $J(\boldsymbol{x}, \boldsymbol{\phi}) = \displaystyle\int_0^\infty \boldsymbol{x}(t)^T\boldsymbol{Q}\boldsymbol{x}(t) + \boldsymbol{\phi}(t)^T\boldsymbol{R}\boldsymbol{\phi}(t)\,\mathrm{d}t$ for a system $\boldsymbol{A}, \boldsymbol{B}$, for some positive semidefinite $\boldsymbol{Q}$ and positive definite $\boldsymbol{R}$.

Optimal Control Law: $\boldsymbol{\phi}^*(t) = -\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x}(t)$ where $\boldsymbol{P}$ solves the algebraic Ricatti equation. The optimal cost is $J(\boldsymbol{x}, \boldsymbol{\phi}^*) = \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}$ where $\boldsymbol{x} = \boldsymbol{x}(0)$.

Algebraic Ricatti Equation: $-\boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{Q} = 0$ which has a unique positive semidefinite solution $\boldsymbol{P}$ when $(\boldsymbol{A}, \boldsymbol{B})$ is stabilizable, and $(\boldsymbol{Q}^{\frac{1}{2}}, \boldsymbol{A})$ is detectable. Note there are multiple solutions, but only one positive semidefinite if conditions are met. If conditions not met, the solution may still exist but not stabilize the system and/or not be optimal.

Matrix Square Root: $\boldsymbol{Q}^{\frac{1}{2}}$ is the matrix such that $(\boldsymbol{Q}^{\frac{1}{2}})^T\boldsymbol{Q}^{\frac{1}{2}} = \boldsymbol{Q}$, which exists and is unique if $\boldsymbol{Q}$ is positive semidefinite. Diagonalize $\boldsymbol{Q} = \boldsymbol{M}^T\boldsymbol{\Lambda}\boldsymbol{M}$ where $\boldsymbol{M}$ is orthogonal (guaranteed by symmetry), then $\boldsymbol{Q}^{\frac{1}{2}} = \boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{M}$.

9 Miscellaneous

Matrix Convergence: $\sum_{k=0}^{\infty} \boldsymbol{M}_k$ converges if $\displaystyle\lim_{n\to\infty}(\boldsymbol{S}_n)_{ij} = a_{ij}, \forall i, j$

Vector Norm: $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ has properties:

1. $\|\boldsymbol{x}\| \geq 0$
2. $\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \boldsymbol{0} \in \mathbb{R}^n$
3. $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$
4. $\|\lambda\boldsymbol{x}\| = |\lambda|\|\boldsymbol{x}\|$

Induced Matrix Norm: $\|\boldsymbol{A}\| = \displaystyle\max_{\|\boldsymbol{x}\|=1} \|\boldsymbol{A}\boldsymbol{x}\|$; same properties as vector norm, plus submultiplicative: $\|\boldsymbol{A}\boldsymbol{B}\| \leq \|\boldsymbol{A}\|\|\boldsymbol{B}\|$

Spectral Theorem: A symmetric $\boldsymbol{P}$ is PD iff all e'vals are positive; PSD iff all e'vals are non-negative. $\boldsymbol{P} = \boldsymbol{P}^T \implies \lambda_i \in \mathbb{R}$ and $\boldsymbol{P}$ diagonalizable.

Inner Product: $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R}$ with the properties:

1. $\langle x, y \rangle = \langle y, x \rangle$
2. $\langle \lambda x_1 + x_2, y \rangle = \lambda\langle x_1, y \rangle + \langle x_2, y \rangle$
3. $\langle x, x \rangle \geq 0$
4. $\langle x, x \rangle = 0 \iff x = \theta$

Cayley-Hamilton Theorem: Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ with characteristic equation $\det(s\boldsymbol{I} - \boldsymbol{A}) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$, then $\boldsymbol{A}^n + a_{n-1}\boldsymbol{A}^{n-1} + \cdots + a_0\boldsymbol{I} = 0$, i.e. every square matrix satisfies its own characteristic equation.

Basis for Null Space: Put in RREF; non-pivot (no leading 1) columns are free (free column count $= \dim(\mathcal{N}(\boldsymbol{A}))$). Find basis by setting one free column at a time to 1, all other free columns to 0, and non-free columns accordingly.

2x2 Inverse: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$