Tutorial 9, Nov 24, 2025

Lyapunov Theorem

- Example: harmonic oscillator $\ddot{y} + \alpha^2 y = 0 \implies \dot{x} = \begin{bmatrix} 0 & 1 \\ -\alpha^2 & 0 \end{bmatrix} x$
  - Eigenvalues are $\pm i\alpha$, so we end up with purely oscillatory solutions that do not decay or grow
  - We can show that this equilibrium is stable; given any ε, we can simply choose $\delta = \varepsilon$, and since the solution amplitude does not grow, we have $\|x\| \leq \varepsilon$ for all time
- For linear systems of the form $\dot{x} = Ax$, we can use a Lyapunov function of the form $V = x^T P x$ where P is symmetric positive definite
  - $\dot{V} = \dot{x}^T P x + x^T P \dot{x} = x^T A^T P x + x^T P A x = x^T (A^T P + PA) x$
  - If we can make P such that $A^T P + PA$ is negative definite, then we can show asymptotic stability (or negative semidefinite for stability)
    * Note that $A^T P + PA$ is already symmetric
  - Theorem: if we have all eigenvalues in the open left half plane, we can show that we can find a symmetric positive definite P such that $A^T P + PA = -\varepsilon I$, for some $\varepsilon > 0$
    * This means that such a system is asymptotically stable in the Lyapunov sense, which matches what we know for linear systems