Tutorial 8, Nov 17, 2025

- Example: point mass with mass m with position $r = (r_x, r_y, r_z)$ on a saddle surface $r_z = r_x^2 - r_y^2$, with gravity acting in $-z$
  - Verify $g(r) = r_z - (r_x^2 - r_y^2)$ is a holonomic constraint and determine the degrees of freedom of this constraint
    * $g(r)$ is scalar, so the number of constraints is $l = 1$
    * To check for holonomic constraint, we check rank $\left(\frac{\partial g}{\partial r}\right) = l$ for all values of r_x, r_y, r_z, i.e. the constraints need to be linearly independent
      - $\frac{\partial g}{\partial r} = \begin{bmatrix} -2r_x & 2r_y & 1 \end{bmatrix}$
      - Since we have 1 in the final component, this has rank 1 for all possible values of the coordinates, so the rank condition is always satisfied
    * The number of degrees of freedom is $3N - 1 = 3 - 1 = 2$ (since r is 3-dimensional and there is only 1 position)
  - Find a set of generalized coordinates and $r = r(q)$
    * Pick $(q_1, q_2) = (r_x, r_y)$
    * This is the best choice since we can easily express r_z in terms of the generalized coordinates; if we chose e.g. r_x, r_z we would need a square root to get r_y
    * $r = r(q) = \begin{bmatrix} q_1 \\ q_2 \\ q_1^2 - q_2^2 \end{bmatrix}$
  - Given an applied force $f_a = \begin{bmatrix} f_x & f_y & 0 \end{bmatrix}^T$, find the generalized force τ
    * $\tau = \left(\frac{\partial r}{\partial q}\right)^T f_a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2q_1 & -2q_2 \end{bmatrix}^T \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$
    * This makes sense since our generalized coordinates are just the normal coordinates but without the z component, so the generalized force is the normal force but truncated
  - Find expressions for the kinetic and potential energy and the Lagrangian
    * $T = \frac{1}{2}m\|\dot{r}\|^2 = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + (2q_1\dot{q}_1 - 2q_2\dot{q}_2))^2 = \frac{1}{2}m(\dot{q}_1^2(1 + 4q_1^2) + \dot{q}_2^2(1 + 4q_2^2) - 8q_1q_2\dot{q}_1\dot{q}_2)$
    * $\mathcal{U} = mgr_z = mg(q_1^2 - q_2^2)$
  - Find all possible virtual displacements on the constraint surface for a mass at $r = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}^T$
    * $\delta r = \frac{\partial r}{\partial q}\,\mathrm{d}q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2q_1 & -2q_2 \end{bmatrix} \begin{bmatrix} \mathrm{d}q_1 \\ \mathrm{d}q_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \mathrm{d}q_1 + \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \mathrm{d}q_2$