Tutorial 10, Dec 1, 2025

- Consider a mass on a pendulum with equation of motion: $ml^2\ddot{\theta} + mgl\sin\theta = \tau$
  - Design a PD controller with gravity compensation to stabilize the system to $(\theta, \dot{\theta}) \to (\pi, 0)$, i.e. pendulum pointing upwards and not moving
    * We can see that the gravity term is $G(\theta) = mgl\sin\theta$ and the inertia term is $D(q) = ml^2$
    * From lecture, the controller is $\tau = mgl\sin\theta + k_p(\pi - \theta) + k_d(0 - \dot{\theta})$ for $k_p > 0, k_d > 0$
  - Show that the equilibrium is asymptotically stable
    * Closed loop system is $ml^2\ddot{\theta} = k_p(\pi - \theta) - k_d\dot{\theta}$
    * Use the Lyapunov function $V(\theta, \dot{\theta}) = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}k_p(\pi - \theta)^2$
      - Note we need to show that this is positive definite
    * $\dot{V} = ml^2\dot{\theta}\ddot{\theta} - k_p(\pi - \dot{\theta})\dot{\theta} = -k_d\dot{\theta}^2$ (substitute $ml^2\ddot{\theta}$)
    * This is only positive semidefinite, so we need to use LaSalle
    * $\dot{V} \equiv 0, \forall t \implies \dot{\theta} \equiv 0 \implies \ddot{\theta} \equiv 0 \implies k_p(\pi - \theta) = 0 \implies \theta = \pi$
- Consider a point mass on a plane, $q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ with mass M; the equations of motion are $M\ddot{q} = u \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$; we want to track the signal $q^r(t) = (t + \sin t, t)$
  - Design a feedback linearization controller
    * Here $M(q) = M, C(q, \dot{q}) = 0, B(q) = 0, G(q) = 0$
    * The system is already linear
    * Choose $u = M\left(\ddot{q}^r + \begin{bmatrix} k_p^1 & 0 \\ 0 & k_p^2 \end{bmatrix}\tilde{q} + \begin{bmatrix} k_d^1 & 0 \\ 0 & k_d^2 \end{bmatrix}\dot{\tilde{q}}\right) = M\begin{bmatrix} -\sin t + k_p^1(t + \sin t - q_1) + k_d^1(1 + \cos t - \dot{q}_1) \\ k_p^2(t - q_2) + k_d^1(1 - \dot{q}_2) \end{bmatrix}$
  - Design a passivity based controller
    * $u = M\left(\begin{bmatrix} -\sin t \\ 0 \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} \dot{q}_1^r - \dot{q}_1 \\ \dot{q}_2^r - \dot{q}_2 \end{bmatrix}\right) + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}\left(\begin{bmatrix} \dot{q}_1^r - \dot{q}_1 \\ \dot{q}_2^r - \dot{q}_2 \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} q_1^r - q_1 \\ q_2^r - q_2 \end{bmatrix}\right)$
  - Compare the two controllers
    * If we choose $k_p^2 = \frac{k_2}{M}\lambda_2, k_d^2 = \lambda_2 + \frac{k_2}{M}, k_p^1 = \begin{bmatrix} k \\ M \end{bmatrix}\lambda_1, k_d^1 = \frac{k_1}{M} + \lambda_1$, then we can see that the two controllers are actually the same
    * This is because we have a linear system