Lecture 6, Sep 15, 2025

Homogeneous Transformations

- Motivation: Suppose we have p^2 and we want to get p^0; we know $p^0 = R_1^0 p^1 + O_1^0$ and $p^1 = R_2^1 p^2 + O_2^1$
  - $p^0 = R_1^0(R_2^1 p^2 + O_2^1) + O_1^0 = R_1^0 R_2^1 p^2 + R_1^0 O_2^1 + O_1^0$
  - The more frames we have, the messier it gets – the algebra is not scalable
  - Can we turn this into a simple matrix multiplication?
- Consider $p^0 = \begin{bmatrix} a & b & c \end{bmatrix}^T$; apply a rigid motion $q^0 = Rp^0 + d^0$ where $R \in SO(3), d^0 \in \mathbb{R}^3$
  - Let the *homogeneous coordinates* be $Q^0 = \begin{bmatrix} q^0 \\ 1 \end{bmatrix}, P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix} \in \mathbb{R}^4$ (capital letters denote homogeneous coordinates)
  - Let the *homogeneous transformation* $H = \begin{bmatrix} R & d^0 \\ 0_{1\times 3} & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}$
  - Notice $HP^0 = \begin{bmatrix} R & d^0 \\ 0_{1\times 3} & 1 \end{bmatrix} \begin{bmatrix} p^0 \\ 1 \end{bmatrix} = \begin{bmatrix} Rp^0 + d^0 \\ 0 + 1 \end{bmatrix} = Q^0$
  - We have turned the entire rigid motion transformation into a single matrix multiplication by H, which is much easier to compose

Definition

$Q^0 = HP^0$ is a *homogeneous transformation*, where

$$P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix}, Q^0 = \begin{bmatrix} q^0 \\ 1 \end{bmatrix}$$

are the *homogeneous coordinates*, formed by adding a 1 after the normal coordinates. H is a member of the *special Euclidean group*

$$SE(3) = \left\{ H = \begin{bmatrix} R & d \\ 0_{1\times 3} & 1 \end{bmatrix} \,\middle|\, R \in SO(3), d \in \mathbb{R}^3 \right\}$$

Each member of $SE(3)$ represents a rigid motion in 3D corresponding to a rotation by R followed by a translation by d.

- For our original problem, let $P^i = \begin{bmatrix} p^i \\ 1 \end{bmatrix}$ be the homogeneous coordinates of p in the 3 frames
  - Let $H_1^0 = \begin{bmatrix} R_1^0 & O_1^0 \\ 0 & 1 \end{bmatrix}$ be the homogeneous transformation from frame 1 to frame 0, and so on for H_2^1 and H_2^0
  - With these definitions, $P^0 = H_1^0 P^1$, $P^1 = H_2^1 P^2$, so $P^0 = H_1^0 P^1 = H_1^0 H_2^1 P^2$
  - If we compute this matrix product, we get the same expression from before, but now we have a much more compact way of representing it
- Similar to the rotations, we define the *elementary homogeneous transformations*:
  - Pure rotations: $\text{Rot}_{x,\alpha} = \begin{bmatrix} R_{x,\alpha} & 0 \\ 0 & 1 \end{bmatrix}$ denotes a rotation by α around x with no translation
    - Similar definitions for $\text{Rot}_{y,\beta}$ and $\text{Rot}_{z,\gamma}$
  - Pure translations: $\text{Trans}_{x,a} = \begin{bmatrix} I & \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$ denotes a translation of a in x with no rotation
    - Similar definitions for $\text{Trans}_{y,b}$ and $\text{Trans}_{z,c}$

Forward Kinematics

- The forward kinematic problem: Given the joint variables $(q_1, \ldots, q_n)$ for an n-joint robot (where q_i denotes the angle for a revolute joint or a length for the prismatic joint), compute the end-effector position and orientation with respect to frame 0
- Consider a manipulator with n links and a base, denoted $0, 1, \ldots, n$ (where 0 is the base), and joints $1, \ldots, n$ (notice the base has no joint); joint i connects link $i-1$ to link i, and is rigidly attached to link $i-1$; when joint i is actuated, link i moves
  - For each link $i \in [1, n]$, we attach a corresponding frame O_i, x_i, y_i, z_i (so this frame moves when joint i moves)
  - Note frame 0 is the inertial frame, which is attached to the fixed base link (link 0)
- Suppose we have all the R_i^{i-1}, O_i^{i-1} relating each frame to the previous frame; let $H_i^{i-1} = \begin{bmatrix} R_i^{i-1} & O_i^{i-1} \\ 0 & 1 \end{bmatrix}$, then the solution of the forward kinematics problem (i.e. the end-effector pose), is $H_n^0 = H_1^0 H_2^1 \cdots H_n^{n-1}$
  - These rotations and translations are functions of the joint variables