Lecture 4, Sep 10, 2025

Rotational Transformations

- Fix a frame O_0, x_0, y_0, z_0 and let $R \in SO(3)$ be a rotation matrix; we will use R to rotate vectors in frame 0
- Consider $w^0 = Rv^0$ – now R is a rotation matrix in frame 0, representing the action of rotating v^0 by some angle
  - This is now a *rotational transformation* in frame 0
    - All rotational transformations (matrices) must have an associated coordinate frame; the matrix operates only on coordinates in that frame
- We can use rotational transformations to generate new frames by rotating the coordinate vectors x, y, z of an existing frame
  - We can define a new frame 1, resulting from rotating each of x_0, y_0, z_0 by R
  - Then $x_1^0 = Rx_0^0$, $y_1^0 = Ry_0^0$ and $z_1^0 = Rz_0^0$ so $R_1^0 = R\begin{bmatrix} x_0^0 & y_0^0 & z_0^0 \end{bmatrix}$
  - But note $x_0^0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, y_0^0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, z_0^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, therefore $R_1^0 = R$
- The rotation matrix of frame 1 with respect to frame 0 (R_1^0) is simply the rotation we apply to the coordinate vectors of frame 0 to get the coordinate vectors of frame 1
- What if we want to represent R in frame 1 instead? If we have $w^0 = Rv^0$, what is R' such that $w^1 = R'v^1$?
  - $w_1 = R_0^1 w^0 = (R_1^0)^T w^0 = (R_1^0)^T Rv^0 = (R_1^0)^T RR_1^0 v^1$
  - Therefore $R' = (R_1^0)^T R(R_1^0)$
  - Recall from linear algebra that this is a similarity transformation used in a change of basis

Composition of Rotational Transformations

- Consider a vector v and 3 frames
- $v^0 = R_2^0 v^2 = R_1^0 v_1 = R_1^0 R_2^1 v^2 \implies R_2^0 = R_1^0 R_2^1$
  - This allows us to compose relative transformations between intermediate frames
  - As a shortcut, notice that the superscript matches the superscript of the left, and the subscript matches the subscript of the right
- Let $R \in SO(3)$ be a rotational transformation in frame 1; we generate frame 2 by rotating frame 1 by R; what is R_2^0?
  - We know $R_2^0 = R_1^0 R_2^1$
  - Since frame 2 is generated by rotating frame 1 by R, $R_2^1 = R$, so $R_2^0 = R_1^0 R$
    - Note the important point part is that R is a rotation in frame 1 – this has to match the superscript on R_2^1
- Now consider the same problem but R is in frame 0 instead (still used to rotate frame 1 to get frame 2); what is R_2^0?
  - $R_2^0 = R_1^0 R_2^1 = R_1^0 (R_1^0)^T RR_1^0 = RR_1^0$
    - R is in frame 0, but we need it in frame 1, so we apply $R_0^1 = (R_1^0)^T$ on the left and R_1^0 on the right
  - Notice that this time the order is reversed!
- Suppose we rotate frame 0 by θ around x_0, then ϕ around z_1 (of the new frame!) to get frame 2; find R_2^0
  - $R_2^0 = R_1^0 R_2^1 = R_{x,\theta} R_{z,\phi}$
  - Note we are able to do this only because the rotations are represented in the respective frame that they are happening to
- Now suppose we rotate frame 0 by θ around x_0, then ϕ around z_0 (old frame), then ψ around z_2; find R_3^0
  - $R_3^0 = R_1^0 R_2^1 R_3^2$
  - $R_1^0 = R_{x,\theta}$ and $R_3^2 = R_{z,\psi}$
  - R_2^1 is $R_{z,\phi}$ in frame 0, so $R_2^1 = (R_1^0)^T R_{z,\phi} R_1^0$

  - Therefore $R_3^0 = R_1^0 R_2^1 R_3^2 = R_1^0 (R_1^0)^T R_{z,\phi} R_1^0 R_{z,\psi} = R_{z,\phi} R_{x,\theta} R_{z,\psi}$
- Let $M_1, M_2, M_3 \in SO(3)$; define frame 1 by rotating frame 0 by M_1; define frame 2 by rotating frame 2 by rotating frame 1 by M_2, expressed in frame 0; define frame 3 by rotating frame 2 by M_3, expressed in frame 1; find R_3^0
  - $R_1^0 = M_1$
  - $R_2^1 = (R_1^0)^T M_2 R_1^0$
  - $R_3^2 = (R_2^1)^T M_3 R_2^1$
  - $R_3^0 = R_1^0 R_2^1 R_3^2$
    $$= R_1^0 R_2^1 (R_2^1)^T M_3 R_2^1$$
    $$= R_1^0 M_3 R_2^1$$
    $$= M_1 M_3 M_1^T M_2 M_1$$