Lecture 34, Dec 1, 2025

Passivity-Based Control With Adaptation

- Now we will attempt to use passivity-based control while learning the system dynamics, resulting in the *Slotine-Li controller*
- Let $r = \dot{\tilde{q}} + \Lambda\tilde{q}, v = \dot{q}^r - \Lambda\tilde{q}, a = \ddot{q}^r - \Lambda\dot{\tilde{q}} = \dot{v}$, where $\tilde{q} = q - q^r$ (note the definition of $\tilde{q}$ is reversed compared to last lecture!)
  - Notice that $\dot{q} = r + v, \ddot{q} = \dot{r} + a$
- Substituting the above: $$M(q)(\dot{r} + a) + C(q, \dot{q})(r + v) + B(q)\dot{q} + G(q) = u$$ $$\implies M(q)\dot{r} + C(q, \dot{q})r = u - (M(q)a + C(q, \dot{q})v + B(q)\dot{q} + G(q))$$
  - The left hand side is the same as what we had for the original passivity-based control, but now we have an extra term on the right
- Recall the linear parametrization: $M(q)a + C(q, \dot{q})v + B(q)\dot{q} + G(q) = Y(q, \dot{q}, a, v)\Theta$, where Y is the known regressor matrix, and Θ is the minimal set of parameters for the system
  - The model becomes $M(q)\dot{r} + C(q, \dot{q})r = u - Y(q, \dot{q}, a, v)\Theta$
- Choose a controller $u = u_s + u_a = -Kr + Y(q, \dot{q}, a, v)\hat{\Theta}$ consisting of a stabilizing and an adaptive term, where $\hat{\Theta}$ is our estimate of the system parameters
  - This results in the closed-loop system $M(q)\dot{r} + (C(q, \dot{q}) + K)r = Y(q, \dot{q}, a, v)\tilde{\Theta}$ where $\tilde{\Theta} = \hat{\Theta} - \Theta$ is the parameter estimation error
- We will apply Lyapunov analysis to see what adaptive control policy gives us an asymptotically stable system, using $V = \frac{1}{2}r^T M(q)r + \tilde{q}^T\Lambda K\tilde{q} + \frac{1}{2}\tilde{\Theta}^T\Gamma^{-1}\tilde{\Theta}$ where Γ, Λ, K are symmetric positive definite matrices
  - This is trivially positive definite at the equilibrium $(r, \tilde{q}, \tilde{\Theta}) = (0, 0, 0)$
  - $$\begin{aligned}\dot{V} &= r^T M(q)\dot{r} + \frac{1}{2}r^T\dot{M}(q, \dot{q})r + 2\tilde{q}^T\Lambda K\dot{\tilde{q}} + \tilde{\Theta}^T\Gamma^{-1}\dot{\tilde{\Theta}} \\ &= r^T(-C(q, \dot{q})r - Kr) + \frac{1}{2}r^T\dot{M}(q, \dot{q})r + 2\tilde{q}^T\Lambda K\dot{\tilde{q}} + r^T Y(q, \dot{q}, a, v)\tilde{\Theta} + \tilde{\Theta}^T\Gamma^{-1}\dot{\tilde{\Theta}} \\ &= -r^T Kr + 2\tilde{q}^T\Lambda K\dot{\tilde{q}} + \tilde{\Theta}^T Y^T(q, \dot{q}, a, v)r + \tilde{\Theta}^T\Gamma^{-1}\dot{\tilde{\Theta}} \\ &= -r^T Kr + 2\tilde{q}^T\Lambda K\dot{\tilde{q}} + \tilde{\Theta}^T\left(Y^T(q, \dot{q}, a, v)r + \Gamma^{-1}\dot{\tilde{\Theta}}\right)\end{aligned}$$
  - This suggests that we should choose $\dot{\hat{\Theta}} = \dot{\tilde{\Theta}} = -\Gamma Y^T r$, then we can make the last term disappear
    - This is the classic gradient law for parameter adaptation
    - Γ is the learning rate, which decides how fast our parameter estimates converges
  - If we do this, then $$\begin{aligned}\dot{V} &= -r^T Kr + 2\tilde{q}^T\Lambda K\dot{\tilde{q}} \\ &= \cdots \text{(see notes from lecture 32)} \\ &= -\begin{bmatrix}\tilde{q}^T & \dot{\tilde{q}}^T\end{bmatrix}\begin{bmatrix}\Lambda K\Lambda & 0 \\ 0 & K\end{bmatrix}\begin{bmatrix}\tilde{q} \\ \dot{\tilde{q}}\end{bmatrix}\end{aligned}$$
- Right now, we have $\dot{V} \leq 0$ so we know the equilibrium is stable, however since our state also includes $\tilde{\Theta}$ this is not fully asymptotically stable
  - By LaSalle, we know that $\dot{V} \to 0$, which means that $(\tilde{q}, \dot{\tilde{q}}) \to (0, 0)$, i.e. we can track the reference
  - However, the parameter estimates themselves do not necessarily converge
- We need a *persistence of excitation* condition on Y to get $\tilde{\Theta} \to 0$
  - Intuitively this means that we need to make the robot do "exciting" behaviours, i.e. behaviours which expose all the parameters of the robot, in order to have the parameters converge
  - Otherwise we can track the reference, but we might not have enough information to fully learn the robot model; e.g. we can command the robot to track a reference signal that only moves a single joint, which means we can't learn about the rest of the system
  - This is a graduate level topic

Review – 2024 Final Q4

- A pendulum has equation of motion $mr^2\ddot{\theta} + k\dot{\theta} + mgr\sin\theta = 0$, which has potential energy $U(\theta) = \frac{g}{r}(1 - \cos\theta)$; either prove that the equilibrium $(\theta^*, \dot{\theta}^*) = (0, 0)$ (i.e. the hanging down position) is asymptotically stable, or stable but not asymptotically stable
  - Let the state $x = \begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}$, then $\dot{x} = \begin{bmatrix} x_2 \\ -\frac{k}{mr^2}x_2 - \frac{g}{r}\sin x_1 \end{bmatrix}$
  - Since this is a mechanical system, we can try using the total energy: $V(x) = \frac{1}{2}x_2^2 + \frac{g}{r}(1 - \cos\theta)$
    - * The first term is a potential energy; nevermind that we don't have m, it doesn't matter
  - We can check that around $(0, 0)$ this is positive definite if we restrict the angle to $(-\pi, \pi)$
  - $\dot{V} = x_2\dot{x}_2 + \frac{g}{r}\sin(x_1)\dot{x}_1$

    $= x_2\left(-\frac{k}{mr^2}x_2 - \frac{g}{r}\sin x_1\right) + \frac{g}{r}\sin(x_1)x_2$

    $= -\frac{k}{mr^2}x_2^2$
  - This is negative semidefinite; by Lyapunov's theorem, this equilibrium is at least stable, but we need LaSalle to conclude that it is asymptotically stable
  - $\dot{V} \equiv 0, \forall t \implies \dot{\theta} = 0 \implies \ddot{\theta} = 0 \implies mgr\sin\theta = 0 \implies \theta = 0$
  - Therefore the equilibrium is asymptotically stable