# Lecture 32, Nov 26, 2025

## Passivity-Based Control

- As usual, consider the augmented robot model with a twice-differentiable reference signal $q^r(t)$ (which may not be constant)
- Choose a controller $u = M(q)\ddot{q}^r + C(q,\dot{q})\dot{q}^r + B(q)\dot{q} + G(q) + K\dot{\tilde{q}}$ where $K$ is symmetric positive definite, $\tilde{q} = q^r - q$, $\dot{\tilde{q}} = \dot{q}^r - \dot{q}$
- This results in the closed-loop system $M(q)\ddot{\tilde{q}} + (C(q,\dot{q}) + K)\dot{\tilde{q}} = 0$
- Let $r = \dot{\tilde{q}}$, so the model becomes $M(q)\dot{r} + (C(q,\dot{q}) + K)r = 0$
  - What if we try the Lyapunov function $V = \frac{1}{2}r^T M(q) r$?
    * $$\begin{aligned}\dot{V} &= r^T M(q)\dot{r} + \frac{1}{2}r^T \dot{M}(q,\dot{q})r \\ &= r^T(-C(q,\dot{q})r - Kr) + \frac{1}{2}r^T\dot{M}r \\ &= -r^TKr + \frac{1}{2}r^T(\dot{M}(q,\dot{q}) - 2C)r \\ &= -r^TKr\end{aligned}$$
    * This is negative definite in $r$, but we still need $\tilde{q}$, so this is still insufficient
- If we define $r = \dot{\tilde{q}} + \Lambda\tilde{q}$, where $\Lambda$ is a diagonal positive definite matrix, then if $r = 0$ for all time, then $\dot{\tilde{q}} + \Lambda\tilde{q} = 0$, which means $\dot{\tilde{q}} = -\Lambda\tilde{q} \implies \tilde{q}(t) = e^{-\Lambda t}\tilde{q}(0)$ which goes to zero
- Choose a new controller $u = M(q)(\ddot{q}^r + \Lambda\dot{\tilde{q}}) + C(q,\dot{q})(\dot{q}^r + \Lambda\tilde{q}) + B(q)\dot{q} + G(q) + K(\dot{\tilde{q}} + \Lambda\tilde{q})$
- The new equations of motion are $M(q)(\ddot{\tilde{q}} + \Lambda\dot{\tilde{q}}) + C(q,\dot{q})(\dot{\tilde{q}} + \Lambda\tilde{q}) + K(\dot{\tilde{q}} + \Lambda\tilde{q}) = 0$ which, using the new definition of $r$, is $M(q)\dot{r} + (C(q,\dot{q}) + K)r = 0$
- Try the Lyapunov function $V = \frac{1}{2}r^TM(q)r + \tilde{q}^TP\tilde{q}$, where $P$ is a symmetric positive definite matrix that is to be determined
  - $V$ is positive definite at $(\tilde{q}, r) = (0,0)$, and $\tilde{q} = 0 \implies r = \dot{\tilde{q}} + \Lambda\tilde{q} = \Lambda\tilde{q}$, so this is equivalent to being positive definite at $(\tilde{q}, \dot{\tilde{q}}) = 0$
  - $$\begin{aligned}\dot{V} &= -r^TKr + 2\tilde{q}^TP\dot{\tilde{q}} \\ &= -(\dot{\tilde{q}} + \Lambda\tilde{q})^TK(\dot{\tilde{q}} + \Lambda\tilde{q}) + 2\tilde{q}^TP\dot{\tilde{q}} \\ &= -\dot{\tilde{q}}^TK\dot{\tilde{q}} - \dot{\tilde{q}}^TK\Lambda\tilde{q} - \tilde{q}^T\Lambda K\dot{\tilde{q}} - \tilde{q}^T\Lambda K\Lambda\tilde{q} + 2\tilde{q}^TP\dot{\tilde{q}} \\ &= -\dot{\tilde{q}}^TK\dot{\tilde{q}} - \tilde{q}^T\Lambda K\Lambda\tilde{q} - 2\tilde{q}^T\Lambda K\dot{\tilde{q}} + 2\tilde{q}^TP\dot{\tilde{q}}\end{aligned}$$
  - Now if we choose $P = \Lambda K$, then $\dot{V} = -\dot{\tilde{q}}^TK\dot{\tilde{q}} - \tilde{q}^T\Lambda K\Lambda\tilde{q} = -\begin{bmatrix}\tilde{q} \\ \dot{\tilde{q}}\end{bmatrix}^T \begin{bmatrix}\Lambda K\Lambda & 0 \\ 0 & K\end{bmatrix}\begin{bmatrix}\tilde{q} \\ \dot{\tilde{q}}\end{bmatrix}$
  - Therefore we have $\dot{V}$ negative definite at the equilibrium, so the closed-loop system is asymptotically stable