Lecture 31, Nov 24, 2025

PD Control With Gravity Compensation

- Again starting with the augmented model, $M(q)\ddot{q} + C(q,\dot{q})\dot{q} + B(q)\dot{q} + G(q) = u$
- Previously, in feedback linearization, we assumed full knowledge of the model; but often this is not realistic, so what if we don't know the full dynamics?
  - In PD control with gravity compensation, we only assume knowledge of $G(q)$, which is much easier to obtain
  - On the other hand, this method only works for a constant reference
- Suppose the reference $q^r(t) \equiv q^r$, i.e. it is constant for all time; choose a controller $u = K_p\tilde{q} + K_d\dot{\tilde{q}} + G(q)$, where $\tilde{q} = q^r - q$ is the tracking error
  - K_p, K_d are symmetric positive definite gain matrices, which are often (but don't have to be) diagonal
- We want to study the closed-loop equilibrium $(\tilde{q}, \dot{\tilde{q}}) = (0,0) \in \mathbb{R}^{2n}$ and show that it is asymptotically stable using Lyapunov and LaSalle
- We will make use of the following facts:
  - $M(q)$ is symmetric positive definite, so it is invertible for all q
  - We can show that $\dot{M}(q,\dot{q}) - 2C(q,\dot{q})$ is skew symmetric
  - $B(q)$ is symmetric positive semi-definite
- The closed-loop system is $M(q)\ddot{q} + C(q,\dot{q})\dot{q} + B(q)\dot{q} = K_p\tilde{q} + K_d\dot{\tilde{q}}$, with gravity terms cancelled
- Take the Lyapunov function $V(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}(q - q^r)^T K_p (q - q^r)$
  - Note that we do not need to include $\dot{\tilde{q}}$ explicitly since it equals $-\dot{q}$ as q^r is a constant; we can also say that the Lyapunov function is a function of $\tilde{q}, \dot{\tilde{q}}$
  - Notice that the first term is the kinetic energy (augmented with motor mass terms)
  - The second term can be thought of as a "virtual potential energy" which pulls the state towards the desired state (recall gravity was cancelled so there is no other source of potential energy)
  - This is clearly positive definite at the equilibrium $(q,\dot{q}) = (q^r, 0)$
- $\dot{V} = \frac{1}{2}\ddot{q}^T M(q)\dot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q,\dot{q})\dot{q} + \frac{1}{2}\dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T K_p(q - q^r) + \frac{1}{2}(q - q^r)^T K_p \dot{q}$

  $= \dot{q}^T M(q)\ddot{q} + (q - q^r)^T K_p \dot{q} + \frac{1}{2}\dot{q}^T M(q,\dot{q})\dot{q}$

  $= \dot{q}^T\left(-C(q,\dot{q})\dot{q} - B(q) + K_p\tilde{q} + K_d\dot{\tilde{q}}\right) - \tilde{q}^T K_p\dot{q} + \frac{1}{2}\dot{q}^T\dot{M}(q,\dot{q})\dot{q}$

  $= \dot{q}^T\left(-C(q,\dot{q})\dot{q} - B(q)\dot{q} + K_d\dot{\tilde{q}}\right) + \frac{1}{2}\dot{q}^T\dot{M}(q,\dot{q})\dot{q}$

  $= \frac{1}{2}\dot{q}^T(\dot{M}(q,\dot{q}) - 2C(q,\dot{q}))\dot{q} - \dot{q}^T B(q)\dot{q} + \dot{q}^T K_d\dot{\tilde{q}}$

  $= -\dot{q}^T B(q)\dot{q} + \dot{q}^T K_d\dot{\tilde{q}}$

  $= -\dot{q}^T B(q)\dot{q} - \dot{q}^T K_d\dot{q}$

  $= -\dot{q}^T(B(q) + K_d)\dot{q}$
  - Note that since $\dot{V}$ is a scalar, all terms are equal to their transpose
  - Because $\dot{M} - 2C$ is skew-symmetric, $\dot{q}^T(\dot{M} - 2C)\dot{q} = \dot{q}^T(\dot{M} - 2C)^T\dot{q} = -\dot{q}^T(\dot{M} - 2C)\dot{q}$ which means the whole term is zero
  - Now because $B(q)$ is positive semidefinite, for any positive definite K_d we have a $\dot{V}$ negative definite in $\dot{q}$ (but not q!)
- We need to apply LaSalle and show that $\dot{V}(t) = 0, \forall t$ forces $q(t) = 0$
  - $\dot{V} = 0 \implies \dot{q}(t) = 0 \implies \ddot{q}(t) = 0$
  - Substituting into the equation of motion for the closed-loop system, $K_p\tilde{q} + K_d\dot{\tilde{q}} = 0$
  - Since K_p is invertible and $\dot{\tilde{q}} = -\dot{q}$, this means $\tilde{q} = 0$ and so $q = q^r$ is the only solution
  - By the LaSalle invariance principle, we conclude that this closed-loop system is asymptotically stable