Lecture 30, Nov 21, 2025

Lyapunov Stability Example and LaSalle Invariance Principle

- Example: dampened mass-spring system with $\dot{x}_1 = x_2, \dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2$, where we take $k = m = b = 1$; prove that the equilibrium $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$ is asymptotically stable
  - For linear systems, we always have a quadratic Lyapunov function
  - Take the Lyapunov function to be $V(x) = x^T P x$ where $P \in \mathbb{R}^{2\times 2}$ is symmetric positive definite
    - For this type of linear system, there is a formula for computing the P matrix
  - Choose $P = \frac{1}{2}\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$, and expanding the Lyapunov function we get $V(x) = \frac{1}{2}(x^2 + x_1x_2 + x_2^2)$
    - We can verify that P is positive definite, which makes $x^T P x$ positive definite at $x = 0$
  - For the derivative, $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = \frac{\partial V}{\partial x} Ax = 2x^T PAx$
  - Expanding this gets us $-\frac{1}{2}(x_1^2 + x_1x_2 + x_2^2) = -V(x)$ which is negative definite since V is positive definite
  - By Lyapunov's theorem, this system is asymptotically stable

Theorem

$P = P^T$ is positive definite if and only if all leading principal minors are positive, where the nth leading principal minor is the determinant of the $n \times n$ sub-matrix formed by taking the first n rows and columns.

- Can we use our intuitive notion of "energy" from physics for this mass-spring system instead?
  - $V(x) = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2$ would be the total energy, consisting of the spring potential energy and kinetic energy
  - Clearly this is positive definite at 0 due to it containing only squared terms
  - $\dot{V}(x) = kx_1\dot{x}_1 + mx_2\dot{x}_2 = kx_1x_2 + mx_2\left(-\frac{k}{m}x_1 - \frac{b}{m}x_2\right) = -bx_2^2$
  - This is negative for $x_2 \neq 0$, but it does not say anything about x_1, so it's not negative definite!

Theorem

LaSalle Invariance Principle: Suppose there exists $V : \mathbb{R}^n \mapsto \mathbb{R}$ positive definite at the equilibrium $\bar{x}$, and $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq 0$, then $\dot{V}(x(t)) \to 0$ as $t \to \infty$.
Furthermore, if $\dot{V}(x) = 0, \forall t \implies x(t) = \bar{x}, \forall t$, then the equilibrium $\bar{x}$ is asymptotically stable.

- The first part says that as we are going down the level sets of V, we are going to either hit zero or reach a point where the derivative is flat and get stuck; the second part says that if the only place we can get stuck forever is at the equilibrium, then we will always end up at the equilibrium
- Returning to our mass-spring example, we can use the LaSalle invariance principle to show that the equilibrium is asymptotically stable
  - If $\dot{V} = 0$ for all t, then $-bx_2(t)^2 = 0 \implies x_2(t) = 0 \implies \dot{x}_2(t) = 0$
  - Recall our equations of motion: $\dot{x}_1 = x_2, \dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2$
  - Therefore we have $-\frac{k}{m}x_1(t) = 0$ from the second equation, so it must be that $x_1(t) = 0$
  - We have shown that $\dot{V} = 0 \implies x = \bar{x}$, so by the LaSalle invariance principle this equilibrium is indeed asymptotically stable
- As with the example, often when we use a physically meaningful Lyapunov function (e.g. potential + kinetic energy), we end up with $\dot{V}$ being only negative semidefinite, so we need to use the LaSalle

invariance principle to make it work