Lecture 3, Sep 8, 2025

Rotations

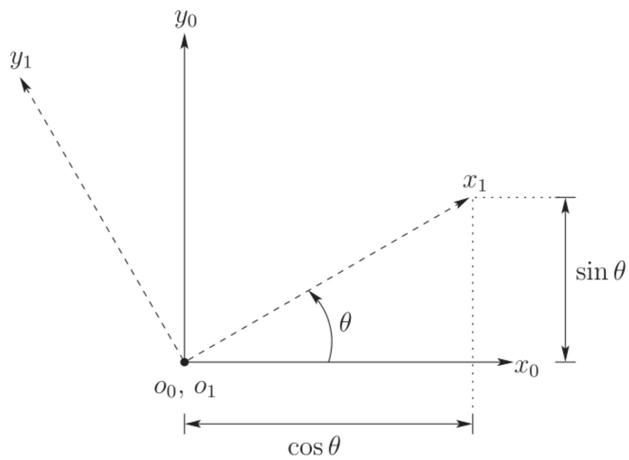

Figure 1: Coordinate frames 1 and 2.

- Consider 2 frames O_0, x_0, y_0 and O_1, x_1, y_1 in $\mathbb{R}^2$ where $O_0 = O_1$; the two frames are separated by an angle θ
- There are 2 ways to describe the relationship between the frames:
  1. Geometric: Specify the angle θ from x_0 to x_1
  2. Algebraic: Specify x_1, y_1 as expressed in frame 0, i.e. x_1^0, y_1^0
     - This is the preferred method that we will explore
- Let $R_1^0 = \begin{bmatrix} x_1^0 & y_1^0 \end{bmatrix}$ be the *rotation matrix* of **frame 1 with respect to frame 0**
  - Since the dot product represents a projection (when normalized), we can obtain the components of x_1^0 by taking the dot product $x_1 \cdot x_0$ and $x_1 \cdot y_0$ to project it onto the axes of frame 0
    * Recall $v_1 \cdot v_2 = \|v_1\|\|v_2\| \cos\theta$, so to actually compute the dot products we can use geometry
  - $x_1^0 = \begin{bmatrix} x_1 \cdot x_0 \\ x_1 \cdot y_0 \end{bmatrix}, y_1^0 = \begin{bmatrix} y_1 \cdot x_0 \\ y_1 \cdot y_0 \end{bmatrix}$
  - Therefore $R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 \end{bmatrix}$
- Generalizing to $\mathbb{R}^3$, $R_1^0 = \begin{bmatrix} x_1^0 & y_1^0 & z_1^0 \end{bmatrix} = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$
  - Notice $R_0^1 = \begin{bmatrix} x_0 \cdot x_1 & y_0 \cdot x_1 & z_0 \cdot x_1 \\ x_0 \cdot y_1 & y_0 \cdot y_1 & z_0 \cdot y_1 \\ x_0 \cdot z_1 & y_0 \cdot z_1 & z_0 \cdot z_1 \end{bmatrix} = (R_1^0)^T$
- Example: Consider the frames as arranged in the figure; find R_1^0
  - By geometry: $x_1^0 = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix}, y_1^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, z_1^0 = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix}$
- Rotation matrices are good for changing vector representations between frames
  - Let $v^1 = \begin{bmatrix} a & b & c \end{bmatrix}^T = ax_1 + by_1 + cz_1$
  - To transform the vector, $v_0 = ax_1^0 + by_1^0 + cz_1^0 = \begin{bmatrix} x_1^0 & y_1^0 & z_1^0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = R_1^0 v^1$
  - Note we used the rotation matrix of frame 1 with respect to frame 0, to transform v^1 to get v^0 – the frame that the rotation matrix is with respect to is the frame we end up getting
    * i.e. The **superscript** on v must match the **subscript** on R

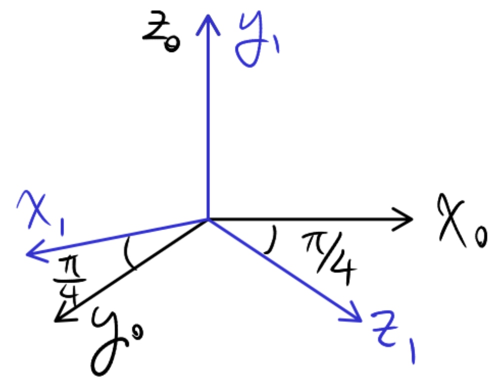

Figure 2: Coordinate frames for the example.

- By the same logic, $v^1 = R_0^1 v^0$, so $R_1^0 = (R_0^1)^{-1} \implies (R_1^0)^T = (R_1^0)^{-1}$
  - To invert a rotation matrix, we can simply take its transpose

Definition

A matrix $R \in \mathbb{R}^{m \times m}$ such that $R^T = R^{-1}$ is an *orthogonal* matrix.

For all orthogonal matrices, all its columns must be unit vectors and are mutually orthogonal (dot product of 0).

- Note that since $1 = \det(I) = \det(RR^T) = \det(R)\det(R^T) = (\det(R))^2 \implies \det(R) = \pm 1$
  - If we choose +1 we get a right-handed coordinate system; similarly -1 gives a left-handed coordinate system
  - We are only interested in right-handed coordinate systems

Definition

The *special orthogonal group* is defined as

$$SO(3) = \{\, R \in \mathbb{R}^{3\times 3} \mid R^T = R^{-1}, \det(R) = 1 \,\}$$

- The *elementary rotation matrices* correspond to rotations by θ about one of the coordinate axes
  - Frame 1 is obtained from frame 0 by rotating by θ about one of the axes; the following matrices give R_1^0
  - $R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
  - $R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
  - $R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$